SOME RESULTS ON COMPLETABILITY IN COMMUTATIVE RINGS

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In this paper, $R$ always denotes a commutative ring with identity. The ideal of nilpotents and the Jacobson radical of the ring $R$ are denoted by $N(R)$ and $J(R)$, respectively. The vector $[a_1, \ldots, a_n]$ is called a primitive row vector provided $1 \in (a_1, \ldots, a_n)$; a primitive row vector $[a_1, \ldots, a_n]$ is called completable provided there exists an $n \times n$ unimodular matrix over $R$ with first row $a_1, \ldots, a_n$. A ring $R$ is called a $B$-ring if given a primitive row vector $[a_1, \ldots, a_n]$, $n \geq 3$, and $(a_1, \ldots, a_{n-2}) \not\in J(R)$, there exists $b \in R$ such that $1 \in (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n)$. Similarly, $R$ is defined to be a Strongly $B$-ring ($SB$-ring), if $d \in (a_1, \ldots, a_n)$, $n \geq 3$, and $(a_1, \ldots, a_{n-2}) \not\in J(R)$ implies that there exists $b \in R$ such that $d \in (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n)$.

In this paper it is proved that every primitive vector over a $B$-ring is completable. It is shown that the following are $B$-rings: $\pi$-regular rings, quasi-semi-local rings, Noetherian rings in which every (proper) prime ideal is maximal, and adequate rings. In addition it is proved that $R[X]$ is a $B$-ring if and only if $R$ is a completely primary ring. It is then shown that the following are $SB$-rings: quasi-local rings, any ring which is both an Hermite ring and a $B$-ring, and Dedekind domains. Finally, it is shown that $R[X]$ is an $SB$-ring if and only if $R$ is a field.

2. $B$-rings.

**Lemma 2.1.** Let $R$ be a ring with $A \subseteq J(R)$, $A$ an ideal of $R$. Then $R$ is a $B$-ring if and only if $R/A$ is a $B$-ring.

**Proof.** Necessity: Let $R$ be a $B$-ring and let

$$(1 + A) \in (a_1 + A, \ldots, a_n + A), n \geq 3$$

and

$$(a_1 + A, \ldots, a_{n-2} + A) \not\subseteq J(R/A) = J(R)/A,$$

where $a_i \in R$, $i = 1, \ldots, n$. Then $1 + A = \sum_{i=1}^n a_i b_i + A$, $b_i \in R$; hence $[a_1, \ldots, a_n]$ is primitive. Since $(a_1, \ldots, a_{n-2}) \not\subseteq J(R)$, it follows that $[a_1 + A, \ldots, a_{n-2} + A, (a_{n-1} + ba_n) + A]$ is primitive for some $b \in R$. Therefore, $R/A$ is a $B$-ring.

Sufficiency: Suppose $R/A$ is a $B$-ring and suppose $[a_1, \ldots, a_n]$ is a $B$-ring. Let $A = [a_1, \ldots, a_n] + A$. Then $1 + A = \sum_{i=1}^n a_i b_i + A$, $b_i \in R$. Hence $[a_1 + A, \ldots, a_n + A, (a_{n-1} + ba_n) + A]$ is primitive for some $b \in R$. Therefore, $R$ is a $B$-ring.
primitive vector with \( (a_1, \ldots, a_{n-2}) \subseteq J(R) \). Hence \([a_1 + A, \ldots, a_n + A]\) is a primitive vector; and, since \((a_1, \ldots, a_{n-2}) \subseteq J(R)\), we have \((a_1 + A, \ldots, a_{n-2} + A) \subseteq J(R/A)\). Since \(R/A\) is a \( B \)-ring, there exists \( b + A \in R/A \) such that \([a_1 + A, \ldots, a_{n-2} + A, (a_{n-1} + ba_n) + A]\) is primitive. It follows that \((1 - u) \in A \subseteq J(R)\), where

\[
u = \sum_{i=1}^{n-2} a_i b_i + (a_{n-1} + ba_n) b_{n-1}, \quad b_i \in R, \quad i = 1, \ldots, n - 1.
\]

Therefore, \( u \) is a unit of \( R \); i.e., \([a_1, \ldots, a_{n-2}, a_{n-1} + ba_n]\) is primitive.

**Theorem 2.1.** If \( R \) is a \( B \)-ring then every primitive row vector over \( R \) is completable.

**Proof.** Let \( R \) be a \( B \)-ring and let \( 1 \in (a_1, \ldots, a_n) \). The theorem clearly holds for \( n = 1 \). If \( n = 2 \), then \( 1 = a_1 x + a_2 y, x, y \in R \) and the matrix \( \begin{pmatrix} a_1 & a_2 \\ -y & x \end{pmatrix} \) is unimodular; hence the result holds for \( n = 2 \).

Let \( n \geq 3 \), and suppose the result is established for \( k < n \).

**Case 1.** If \((a_1, \ldots, a_{n-2}) \subseteq J(R)\) and \( 1 = \sum_{i=1}^{n-2} a_i b_i, b_i \in R \), then \( 1 - \sum_{i=1}^{n-2} a_i b_i = a_{n-1} b_{n-1} + a_n b_n \) is a unit \( u \in R \). Let

\[
V = \begin{pmatrix}
a_{n-1} & a_n & a_1 & a_2 & \cdots & a_{n-2} \\
-b_n & b_{n-1} & 0 & 0 & \cdots & 0 \\
O & I^{n-2}
\end{pmatrix}.
\]

Then \( V \) has determinant \( u \), and it follows that \([a_1, \ldots, a_n]\) is completable.

**Case 2.** If \((a_1, \ldots, a_{n-2}) \nsubseteq J(R)\), then \( 1 \in (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n) \), for some \( b \in R \). By the induction hypothesis, \([a_1, \ldots, a_{n-2}, a_{n-1} + ba_n]\) is completable to an \((n - 1) \times (n - 1)\) unimodular matrix \( D \). Let

\[
U = \begin{pmatrix} I^{n-2} & 0 & 0 \\
0 & \cdots & 1 & 0 \\
0 & \cdots & -b & 1
\end{pmatrix}
\]

and let \( B = \begin{pmatrix} D & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix} \).

Then \( BU \) is an \( n \times n \) unimodular matrix whose first row is \([a_1, \ldots, a_n]\).

For convenience, we introduce the notation \( Z(A) \) to mean the set of maximal ideals containing the ideal \( A \); \( Z(a) \) will denote the set of maximal ideals containing the element \( a \).

**Theorem 2.2** If \( R \) is a ring such that for every ideal \( A \nsubseteq J(R) \), \( Z(A) \) is finite, then \( R \) is a \( B \)-ring.
Proof. The essentials of the proof are due to Reiner [4]. Let $1 \in (a_1, \cdots, a_n)$, $n \geq 3$, and $(a_1, \cdots, a_{n-2}) \not\subseteq J(R)$. By the hypothesis on $R$, $Z(A)$ is finite where $A = (a_1, \cdots, a_{n-2})$. Let $Z(A) = \{M_1, \cdots, M_r\}$, and note that if $b \in R$ and $a_{n-1} + ba_n \in M_i$, $i = 1, \cdots, r$, then $[a_1, \cdots, a_{n-2}, a_{n-1} + ba_n]$ is primitive.

For any $M_i \in Z(A)$ such that $a_n \in M_i$, we have $a_{n-1} + ba_n \in M_i$, for all $b \in R$; otherwise, $a_{n-1} \in M_i$, and $(a_1, \cdots, a_n) \subseteq M_i$ which contradicts the hypothesis that $[a_1, \cdots, a_n]$ is primitive.

For those $M_i \in Z(A)$ for which $a_n \in M_i$, we have $(a_n, M_i) = (1)$. Hence there exists an $x_i$ such that $a_n x_i \equiv a_{n-1} \pmod{M_i}$. For these $M_i$, we can find (by the Chinese Remainder Theorem) an element $b \in R$ such that $b \equiv 1 - x_i \pmod{M_i}$. It follows that $a_{n-1} + ba_n \in M_i$, $i = 1, \cdots, r$. Hence $[a_1, \cdots a_{n-2}, a_{n-1} + ba_n]$ is primitive.

It follows from this theorem that quasi-semi-local rings and Noetherian rings in which every proper prime ideal is maximal (in particular, Dedekind domains) are $B$-rings.

**Lemma 2.2.** Let $R$ be an $F$-ring (i.e., a ring in which every finitely generated ideal is principal) which satisfies the condition that if $1 \in (a_1, a_2, a_3)$ with $a_1 \in J(R)$ then $1 \in (a_1, a_2 + ba_3)$ for some $b \in R$. Then $R$ is a $B$-ring.

Proof. Let $1 \in (a_1, \cdots, a_n)$, $n \geq 3$, and let $(a) = (a_1, \cdots, a_{n-2}) \not\subseteq J(R)$. Hence $1 \in (a, a_{n-1}, a_n)$. By the hypothesis on $R$, $1 \in (a, a_{n-1} + ba_n)$; hence, $R$ is a $B$-ring.

**Theorem 2.3.** If $R$ is an $F$-ring which satisfies the condition that for every $a, c \in R$ with $a \in J(R)$, there is an $r \in R$ such that $Z(r) = Z(a) - Z(c)$, then $R$ is a $B$-ring.

Proof. The proof is essentially the same as the proof of Theorem 5 of [2]. Let $1 \in (a, b, c)$, $a \in J(R)$. By the hypothesis on $R$ there exists $r \in R$ such that $Z(r) = Z(a) - Z(c)$. Hence $(c, r) = (1)$, so there exists $q \in R$ such that $1 \in (r, b + qc)$. We claim $(a, b + qc) = (1)$. Otherwise, there exists a maximal ideal $M$ of $R$ such that $(a, b + qc) \subseteq M$. Hence $M \in Z(a)$ and $M \in Z(b + qc)$. Since $1 \in (r, b + qc)$ it follows that $M \in Z(r)$, so $M \in Z(c)$. But we now have $M \in Z(b)$, contrary to $(a, b, c) = (1)$. Therefore $(a, b + qc) = (1)$. Lemma 2.2 completes the proof.

**Theorem 2.4.** Every adequate ring is a $B$-ring.

Proof. In the proof of Theorem 5.3 of [3], Kaplansky shows that
if $R$ is an adequate ring and if $1 \in (a, b, c), a \neq 0$, then there exists $q \in R$ such that $1 \in (a, b + qc)$. Since an adequate ring is an $F$-ring, the result follows from Lemma 2.2.

**Theorem 2.5.** Every $\pi$-regular ring is a $B$-ring.

**Proof.** If $R$ is a $\pi$-regular ring, and if $a \in R/N(R)$, then by Lemma 2.2 of [5], $a$ is an associate of $e + \beta$, $e$ an idempotent and $\beta$ a nilpotent of the $\pi$-regular ring $R/N(R)$. Since $\beta = 0$, $a = ve$, $v$ a unit of $R/N(R)$. Therefore, $a^2 = ve$ and $v^{-1}a^2 = ve = a$. Hence, $R/N(R)$ is a regular ring and therefore an adequate ring ([1, Th. 11]). Theorem 2.4 and Lemma 2.1 complete the proof.

**Theorem 2.6.** Let $D$ be an integral domain, $K$ its quotient field. Let $R = \{(a_1, \ldots, a_k, a, a, \ldots): a_i \in K, a \in D\}$, where $k$ is a nonnegative integer ($k$ may be different for distinct elements of $R$). The operations in $R$ are component-wise addition and multiplication. If $R$ is a $B$-ring then $D$ is a $B$-domain.

We illustrate the proof. Suppose $R$ is a $B$-ring and let $1 \in (a, b, c)$, $a, b, c \in D$, $1 = aa' + bb' + cc'$. Let $\hat{a} = (1, a, a, \ldots), \hat{b} = (0, b, b, \ldots), \hat{c} = (0, c, \ldots), \hat{a}' = (1, a', a', \ldots), \hat{b}' = (0, b', b', \ldots), \hat{c}' = 0, c', c', \ldots)$. Then $1 = \hat{a}\hat{a}' + \hat{b}\hat{b}' + \hat{c}\hat{c}'$. If $\hat{a} \in J(R)$, then $\hat{1} - \hat{a} = (0, 1 - a, 1 - a, \ldots)$ is a unit of $R$. Since this is false, $\hat{a} \in J(R)$, hence $\hat{1} \in (\hat{a}, \hat{b} + \hat{y}\hat{c})$ for some $\hat{y} \in R$. Therefore $\hat{1} = \hat{a}\hat{d} + (\hat{b} + \hat{y}\hat{c})\hat{e}$, where $\hat{d}, \hat{c}, \hat{e} \in R$. Let $\hat{d} = (d_1, \ldots, d_p, d, d, \ldots), \hat{e} = (e_1, \ldots, e_q, e, e, \ldots), \hat{y} = (y_1, \ldots, y_r, y, y, \ldots)$ and let $\lambda = \max(1, p, q, r)$. In the $(\lambda + 1)$st entry of $\hat{a}\hat{d} + (\hat{b} + \hat{y}\hat{c})\hat{e}$, we have $ad + (b + yc)e$; i.e., $1 \in (a, b + yc)$. Hence, $D$ is a $B$-domain.

**Theorem 2.7.** $R[X]$ is a $B$-ring if and only if $R$ is a completely primary ring.

**Proof.** Sufficiency: Let $R$ be a completely primary ring. Since $R/N(R)$ is a field and since $(R/N(R))[X] \cong R[X]/N(R)[X]$, it follows from Theorem 2.2 that $R[X]/N(R)[X]$ is a $B$-ring. Since $N(R)[X] = N(R[X])$, the result follows from Lemma 1.2.1.

Necessity: Assume that $R$ is not completely primary and that $R[X]$ is a $B$-ring. Let $r$ be a nonunit, nonnilpotent element of $R$. Then $1 \in (r, 1 + X, X^2)$ and $r \in J(R[X])$. By the assumption that $R[X]$ is a $B$-ring, we have $1 \in (r, 1 + X + X^2\tilde{f}(X))$ for some $\tilde{f}(X) \in R[X]$. Let $\bar{a}$ denote the image of $a \in R$ under the natural homomorphism of $R[X]$ onto $(R/rR)[X]$. Then $1 \in (0, 1 + X + X^2\tilde{f}(X))$ and $1 + X + X^2\tilde{f}(X)$ is a unit of $(R/rR)[X]$. This is a contradiction since the coefficient of $X$ is not nilpotent.
Since $R[X]$ cannot be completely primary, (clearly, $X$ is neither a unit nor a nilpotent) it follows that for every ring $R$, $R[X, Y] = R[X][Y]$ is not a $B$-ring.

3. Strongly $B$-rings. We now turn our attention to the study of SB-rings. Our main objective here is to compare the theory of this particular subclass of $B$-rings with that of $B$-rings given in the last section.

**Lemma 3.1.** $R$ is an SB-ring if and only if for every $s, c_1, c_2, c_3 \in R$ with $s \in (c_1, c_2, c_3)$ and $c_1 \in J(R)$, it follows that $s \in (c_1, c_2 + bc_3)$ for some $b \in R$.

**Proof.** The necessity clearly follows from the definition of an SB-ring.

Sufficiency: Let $r \in (a_1, \ldots, a_n), n \geq 3$, with $(a_1, \ldots, a_n) \nsubseteq J(R)$. Without loss of generality, we may assume that $a_{n-2} \in J(R)$. Suppose $r = \sum_{i=1}^{n} a_i x_i$ and let $s = a_{n-2} x_{n-2} + a_{n-1} x_{n-1} + a_n x_n$. Then $r \in (a_1, \ldots, a_{n-3}, s)$ and $s \in (a_{n-2}, a_{n-1}, a_n)$. Since $a_{n-2} \in J(R), s \in (a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$. Therefore $r \in (a_1, \ldots, a_{n-3}, s) \subseteq (a_1, \ldots, a_{n-2} + ba_n)$, and the proof is complete.

In view of Lemma 3.1, we need only consider triples instead of arbitrary $n$-tuples in our study of SB-rings.

**Lemma 3.2.** The homomorphic image of an SB-ring is an SB-ring.

**Proof.** Let $\bar{R}$ be the image of $R$ under the homomorphism $\phi$, and let $\bar{d} \in (\bar{a}_1, \bar{a}_2, \bar{a}_3)$ with $\bar{a}_1 \in J(\bar{R}), \bar{a}_2, \bar{a}_3, \bar{d} \in \bar{R}$. Suppose $\bar{d} = \sum_{i=1}^{3} \bar{a}_i \bar{x}_i, \bar{x}_i \in \bar{R}$ and let $a_i \phi = \bar{a}_i, x_i \phi = \bar{x}_i, i = 1, 2, 3$. Let $d = \sum_{i=1}^{3} a_i x_i$. Since $(J(R)) \phi \subseteq J(\bar{R})$, we have $a_i \in J(R)$; hence, $d \in (a_1, a_2 + ba_n)$ for some $b \in R$. Since $d \phi = \bar{d}$, we have $\bar{d} \in (\bar{a}_1, \bar{a}_2 + \bar{b} \bar{a}_n)$, where $b \phi = \bar{b}$. Hence $\bar{R}$ is an SB-ring.

**Theorem 3.1.** Every quasi-local ring is an SB-ring.

**Proof.** Let $d \in (a_1, a_2, a_3)$, with $a_1 \in J(R), R$ a quasi-local ring. Since $a_1 \in J(R), a_1$ is a unit of $R$; hence, $d \in (a_1, a_2 + ba_n) = (1)$ for every $b \in R$.

**Lemma 3.3.** Let $A = (a_1, \ldots, a_n), n \geq 3$, be an ideal in a Dedekind domain $R$. If $B = (a_1, \ldots, a_{n-1}) \neq (0)$, then $A = (a_1, \ldots, a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$. 

Proof. Let \( A = \prod_{i=1}^{t} M_{i}^{n_{i}} \) and let \( B = \prod_{i=1}^{t} M_{i}^{n_{i}} \) be the representations of the ideals \( A \) and \( B \) as a product of powers of distinct maximal ideals. Since \( B \subseteq A \), we may order the \( M_{i} \) so that \( 0 \leq \alpha_{i} < \beta_{i} \) for \( 1 \leq i \leq r \), and \( \alpha_{i} = \beta_{i} \) for \( r + 1 \leq i \leq t \). Let \( 1 \leq k \leq r \). We claim that either \( a_{n_{i}} - b_{n_{i}} \) or \( a_{n_{i}} \) does not belong to \( M_{i}^{n_{i}+1} \). For suppose both \( a_{n_{i}} - b_{n_{i}} \) and \( a_{n_{i}} \) belong to \( M_{i}^{n_{i}+1} \). Then \( A \nsubseteq M_{i}^{n_{i}+1} \), a contradiction. Since the \( M_{i} \) are relative prime, the Chinese Remainder Theorem guarantees the existence of a \( b \in R \) satisfying:

\[
b \equiv 0 \pmod{M_{i}^{n_{i}+1}} \quad \text{if} \quad a_{n_{i}} - b_{n_{i}} \notin M_{i}^{n_{i}+1} \\
b \equiv 1 \pmod{M_{i}^{n_{i}+1}} \quad \text{if} \quad a_{n_{i}} \notin M_{i}^{n_{i}+1}
\]

for \( k = 1, 2, \ldots, r \). It follows that \( a_{n_{i}} + ba_{n_{i}} \notin M_{i}^{n_{i}+1} \) for \( k = 1, 2, \ldots, r \). Let \( (a_{1}, \ldots, a_{n_{2}}, a_{n_{1}} + ba_{n_{1}}) = \prod_{i=1}^{t} M_{i}^{n_{i}} \). Since \( (a_{1}, \ldots, a_{n_{2}}, a_{n_{1}} + ba_{n_{1}}) \subseteq A = \prod_{i=1}^{t} M_{i}^{n_{i}} \), it follows that \( \mu_{i} \geq \alpha_{i} \), \( i = 1, 2, \ldots, t \). Since \( B = \prod_{i=1}^{t} M_{i}^{n_{i}} \subseteq \prod_{i=1}^{t} M_{i}^{n_{i}} \subseteq \prod_{i=1}^{t} M_{i}^{n_{i}} = A \), and since \( \beta_{i} = \alpha_{i} \), \( r + 1 \leq i \leq t \), it follows that \( \mu_{i} = \beta_{i} = \alpha_{i} \), \( r + 1 \leq i \leq t \). If \( \mu_{i} > \alpha_{i} \) for some \( i \) with \( 1 \leq i \leq r \), then \( a_{n_{i}} + ba_{n_{i}} \notin M_{i}^{n_{i}} \subseteq M_{i}^{n_{i}+1} \), a contradiction. Hence, \( \mu_{i} = \alpha_{i} \), \( i = 1, 2, \ldots, t \). Equivalently, \( (a_{1}, \ldots, a_{n_{2}}, a_{n_{1}} + ba_{n_{1}}) = A \).

As an immediate consequence, we have:

**THEOREM 3.2.** A Dedekind domain is an SB-ring.

**LEMMA 3.4.** Let \( R \) be a B-ring, let \( e = e^{2} \in R \), and let \( e \in (a_{1}, \ldots, a_{n}) \) with \( (a_{1}, \ldots, a_{n}) \notin J(R) \), \( n \geq 3 \). Then \( e \in (a_{1}, \ldots, a_{n-2}, a_{n-1} + ba_{n}) \) for some \( b \in R \).

**Proof.** Since the case \( e = 1 \) is covered by the hypothesis, we may assume \( e \neq 1 \). Let \( e = \sum_{i=1}^{n} a_{i}x_{i} = \sum_{i=1}^{n} (a_{i}e)(x_{i}e) \). Hence, \( 1 = (a_{e} + 1 - e)(x_{e} + 1 - e) + \sum_{i=2}^{n} (a_{i}e)(x_{i}e) \). Thus,

\[
1 \in (a_{e} + 1 - e, a_{e}x_{i}, \ldots, a_{e}e).
\]

If \( a_{e} + 1 - e \in J(R) \), then \( 1 - (a_{e} + 1 - e) = e(1 - a_{i}) \) is a unit of \( R \), a contradiction since \( e = e^{2} \), \( e \neq 1 \). Thus, since \( R \) is a B-ring, we have \( 1 \in (a_{e} + 1 - e, a_{e}x_{i}, \ldots, a_{e}^{n}e, a_{n-1}e + ba_{n}e) \) for some \( b \in R \). Therefore, \( e \in (a_{e} + 1 - e, a_{e}x_{i}, \ldots, a_{n-1}e, a_{n-1}e + ba_{n}e) \subseteq (a_{1}, a_{2}, \ldots, a_{n-2}, a_{n-1} + ba_{n}) \).

**COROLLARY.** If \( R \) is a regular ring then \( R \) is an SB-ring.

**Proof.** The result is immediate from Theorem 2.5 and Lemma 3.4; since, for every \( r \in R \), \( r \) is an associate of some idempotent \( e \in R \) ([1, Lemma 10]).

**THEOREM 3.3.** If a B-ring \( R \) is also an Hermite ring, then \( R \) is an SB-ring.
Proof. Let \( d \in (a_1, a_2, a_3) = (a), a \in J(R) \). By Corollary 5 of [1], there exist \( b_1, b_2, b_3 \) such that \( a_1 = b_1a, a_2 = b_2a, a_3 = b_3a, \) and \( (b_1, b_2, b_3) = (1) \). Since \( R \) is a \( B \)-ring and since \( b_i \in J(R) \), there exists a \( q \in R \) such that \( (b_1, b_2 + qb_3) = (1) \). Therefore, \( (a) = (b_1a, b_2a + qb_3a) = (a_1, a_2 + qa_3) \). Hence, \( d \in (a_1, a_2 + qa_3) \).

**Corollary.** Every adequate domain is an \( SB \)-ring.

**Proof.** An adequate domain is both an \( F \)-domain and a \( B \)-ring. Since every \( F \)-domain is an Hermite ring, the result follows from Theorem 3.3.

**Corollary.** If \( R \) is an \( F \)-ring with infinitely many maximal ideals and, if for every ideal \( A \subseteq J(R), Z(A) \) is finite, then \( R \) is an \( SB \)-ring.

**Proof.** \( R \) is necessarily a \( B \)-ring by Theorem 2.2. By the proof of Corollary 2 of [2], \( R \) is also an Hermite ring. Theorem 3.3 completes the proof.

**Theorem 3.4.** \( R[X] \) is an \( SB \)-ring if and only if \( R \) is a field.

**Proof.** The sufficiency follows from Theorem 3.2. To prove the necessity, let \( r \in R, r \neq 0 \). Then \( r \in (X^2, X, r) \) and \( X^2 \in J(R[X]) \). If \( R[X] \) is an \( SB \)-ring then \( r \in (X^2, X + rb(X)) \) for some \( b(X) \in R[X] \). Let \( r = X^2f(X) + (X + rb(X))g(X) \), where \( f(X) \) and \( g(X) \in R[X] \), and let \( f_i, g_i, b_i \) represent the coefficient of \( X^i \) in the polynomials \( f(X), g(X), b(X) \), respectively. Equating coefficients in the above equation gives \( r = rb_0g_0 \) and \( 0 = g_0 + r(b_0g_1 + g_0b_1) \). Hence \( r \) divides \( g_0 \) and therefore \( r = r^2k \) for some \( k \in R \). Hence \( rk = (rk)^2 \); therefore, \( rk \) is an idempotent of \( R \). Since \( R[X] \) is a \( B \)-ring, \( R \) must be a completely primary ring by Theorem 2.7. It follows that the idempotent \( rk \) is either 0 or 1. Since \( rk = 0 \) and \( r = r^2k \) imply \( r = 0 \), we conclude that \( rk = 1 \); i.e., \( r \) is a unit of \( R \). Hence, \( R \) is a field and the proof is complete.

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