VON NEUMANN ALGEBRAS GENERATED BY OPERATORS SIMILAR TO NORMAL OPERATORS

WARREN R. WOGEN
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A normal operator generates an abelian von Neumann algebra. However, an operator which is similar to a normal operator may generate a von Neumann algebra which is not even type I. In fact, it is shown that if $\mathcal{A}$ is a von Neumann algebra on a separable Hilbert space and $\mathcal{A}$ has no type II finite summand, then $\mathcal{A}$ has a generator which is similar to a self-adjoint and $\mathcal{A}$ has a generator which is similar to a unitary. The restriction that $\mathcal{A}$ have no type II finite summand can be removed provided that it is assumed that every type II finite von Neumann algebra has a single generator.

Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{A}$ be a von Neumann algebra on $\mathcal{H}$. $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. For $n \geq 2$, let $M_n(\mathcal{A})$ denote the von Neumann algebra of $n \times n$ matrices with entries in $\mathcal{A}$. If $T$ is a bounded operator, the $\mathcal{B}(T)$ is the von Neumann algebra generated by $T$.

We begin with some lemmas.

**Lemma 1.** Let $\mathcal{A} = \mathcal{B}(C)$ and suppose $n \geq 3$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be sequences of complex numbers such that the $\lambda_k$ are distinct, each $\lambda_k \neq 0$, and $\|C\| < 1/2$. Define $A = (A_{i,j})_{i,j=1}^n \in M_n(\mathcal{A})$ by $A_{i,i} = \lambda_i I$, $A_{i,i+1} = a_{i} I$, $A_{i,j} = 0$ otherwise. Define $B = (B_{i,j})_{i,j=1}^n \in M_n(\mathcal{A})$ by $B_{i,i} = \lambda_i I$ and $B_{i,j} = 0$ if $i \neq j$. Then $A$ and $B$ are similar, and $\mathcal{B}(A) = M_n(\mathcal{A})$.

**Proof.** It follows from [11, Lemma 1] that $\mathcal{B}(A) = M_n(\mathcal{A})$. To show that $A$ and $B$ are similar we need only that the $\lambda_k$ are distinct. We must find an invertible operator $S$ such that $AS = SB$. Such an $S$ of the form $S = I + N$, where $N$ is lower triangular and nilpotent, can be computed easily. Merely perform the matrix multiplications and solve for the entries of $S$. We omit the details.

**Remark 1.** If the operator $S = I + N$ in Lemma 1 is computed, we see that we can make the entries of $N$ small by choosing $\|C\|$, $|a_1|$, $|a_2|$, $\cdots$, $|a_{n-1}|$ suitably small. Hence we can suppose that $\|N\| < 1/2$. Then $\|S\| = \|I + N\| < 3/2$ and $\|S^{-1}\| = \|I - N + N^2 - \cdots \pm N^{s-1}\| < 2$. Note also that by choosing $\|C\|$, $|a_1|$, $|a_2|$, $\cdots$, $|a_{n-1}|$ suitably, we can assume that $\|A\| \leq \|B\| + 1$. 

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The following is a corollary of Lemma 1.

**COROLLARY 1.** If $\mathcal{A}$ is a properly infinite von Neumann algebra on $\mathcal{H}$, then $\mathcal{A}$ has a generator which is similar to a self-adjoint operator.

**Proof.** If $\mathcal{A}$ is properly infinite, then it is well-known that $\mathcal{A}$ is *-isomorphic to $M_3(\mathcal{H})$. $\mathcal{A}$ has a single generator $C$ by [10]. Construct a generator $A$ of $M_3(\mathcal{H})$ as in Lemma 1, with $\lambda_1$, $\lambda_2$, and $\lambda_3$ real. Then $A$ is similar to self-adjoint operator by Lemma 1. (Another easy proof of Corollary 1 can be deduced from methods in the proof of Corollary 1 in [1].)

It has been shown that if $\mathcal{A}$ is properly infinite, then $\mathcal{A}$ is generated by three projections [9] and by two idempotents [4]. A related result is

**COROLLARY 2.** If $\mathcal{A}$ is a properly infinite von Neumann algebra on $\mathcal{H}$, then $\mathcal{A}$ is generated by three commuting idempotents.

**Proof.** If $A$ is the generator of $\mathcal{A}$ constructed in Corollary 1, let $E$ be the (idempotent valued) spectral measure of $A$. Then $E(\lambda_1)$, $E(\lambda_2)$, and $E(\lambda_3)$ are the required commuting idempotents.

Let $\sigma(C)$ denote the spectrum of the operator $C$.

**LEMMA 2.** Let $\mathcal{A} = \mathcal{R}(C)$. Let
\[
A = \begin{bmatrix} C & 0 \\ aI & \lambda I \end{bmatrix}, \quad B = \begin{bmatrix} C & 0 \\ 0 & \lambda I \end{bmatrix},
\]
where $a \neq 0$ and $\lambda \in \sigma(C)$. Then $A$ is similar to $B$, and $\mathcal{R}(A) = M_3(\mathcal{H})$.

**Proof.** A routine computation shows that
\[
\mathcal{R}(A)' = \left\{ \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} : T \in \mathcal{H} \right\}.
\]
It follows that $\mathcal{R}(A) = \mathcal{R}(A)'' = M_3(\mathcal{H})$. Let
\[
S = \begin{bmatrix} I & 0 \\ a(C - \lambda I)^{-1} & I \end{bmatrix}.
\]
Then $S$ is invertible and $AS = SB$.

**LEMMA 3.** Let $\{A_k\}_{k=0}^{\infty}$ be a uniformly bounded sequence of operators. Suppose that the $A_k$ have pairwise disjoint spectra. Then
\( \mathcal{R} \left( \bigoplus_{k=0}^{\infty} A_k \right) = \bigoplus_{k=0}^{\infty} \mathcal{R}(A_k) \).

**Proof.** The proof given here is due essentially to Rosenthal [8, Th. 3]. (See also [3, Lemma].) Let \( A = \sum_{k=0}^{\infty} \bigoplus A_k \). Suppose \( C = (C_{i,j})_{i,j=0}^{\infty} \) commutes with \( A \). Then
\[
C_{i,j}A_j = A_iC_{i,j} \quad \text{for all } i,j.
\]
If \( i \neq j \), then \( \sigma(A_i) \) and \( \sigma(A_j) \) are disjoint, so by a theorem of Rosenblum [7], \( C_{i,j} = 0 \). It follows that \( \mathcal{R}(A) = \sum_{k=0}^{\infty} \bigoplus \mathcal{R}(A_k)' \), so that \( \mathcal{R}(A) = \mathcal{R}(A)' = \sum_{k=1}^{\infty} \bigoplus \mathcal{R}(A_k) \).

**Theorem 1.** If \( \mathcal{A} \) is a von Neumann algebra on a separable Hilbert space such that \( \mathcal{A} \) has no type II finite summand, then \( \mathcal{A} \) has a generator which is similar to a self-adjoint operator.

**Proof.** Write \( \mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n \), where \( \mathcal{A}_0 \) is properly infinite and for each \( n \geq 1 \), \( \mathcal{A}_n \) is an \( n \)-homogeneous type I summand (see [2]). (Note that some of these summands may be absent.) Let \( \{I_n\}_{n=0}^{\infty} \) be a pairwise disjoint sequence of nonempty subintervals of \([0,1]\).

By Corollary 1, we can choose \( A_0 \) and an invertible operator \( S_0 \) such that \( \mathcal{R}(A_0) = \mathcal{A}_0, S_0 A_0 S_0^{-1} \) is self-adjoint, and \( \sigma(A_0) \subset I_0 \).

For each \( n \geq 1 \), \( \mathcal{A}_n \) is \(*\)-isomorphic to \( M_n(\mathbb{C}_n) \), where \( \mathbb{C}_n \) is the center of \( \mathcal{A}_n \) (see [2]). \( \mathbb{C}_n \) is abelian, so \( \mathbb{C}_n \) has a self-adjoint generator by [5]. Let \( A_i \) be a self-adjoint generator of \( \mathcal{A}_i = \mathbb{C}_i \). By translating and scaling, if necessary, we can assume \( \sigma(A_i) \subset I_i \). Let \( S_i \) be the identity in \( \mathcal{A}_i \).

Let \( C \) be a self-adjoint generator of \( \mathcal{C}_2 \) with \( \sigma(C) \) properly contained in \( I_2 \). Let \( \lambda \in I_2 \) with \( \lambda \in \sigma(C) \). Let \( a \neq 0 \) and let
\[
A_2 = \begin{bmatrix} C & 0 \\ aI & \lambda I \end{bmatrix}.
\]

Then by Lemma 2, \( \mathcal{R}(A_2) = \mathcal{A}_2 \) and for some invertible \( S_2, S_2 A_2 S_2^{-1} \) is self-adjoint. Also, \( \sigma(A_2) = \sigma(C) \cup \{\lambda\} \subset I_2 \).

For \( n \geq 3 \), use Lemma 1 to construct \( A_n \) and an invertible \( S_n \) such that \( \mathcal{R}(A_n) = \mathcal{A}_n, S_n A_n S_n^{-1} \) is self-adjoint, and \( \sigma(A_n) \subset I_n \). Moreover by Remark 1, we can suppose that the sequences \( \{A_n\}, \{S_n\} \), and \( \{S_n^{-1}\} \) are uniformly bounded.

Let \( A = \sum_{n=0}^{\infty} \bigoplus A_n \), and let \( S = \sum_{n=0}^{\infty} S_n \). Then \( A \) and \( S \) are bounded operators, \( S \) is invertible, and \( SAS^{-1} \) is self-adjoint. Finally \( \mathcal{R}(A) = \sum_{n=0}^{\infty} \bigoplus A_n \) by Lemma 3.

It has long been conjectured that every von Neumann algebra on a separable Hilbert space has a single generator. Results in [6] and
reduce the proof of the conjecture to showing that (S) Every type II finite von Neumann algebra on a separable Hilbert space has single generator. (See [4] for a partial solution to this conjecture.)

**Theorem 2.** If (S) is true and \( \mathcal{A} \) is a von Neumann algebra on a separable Hilbert space, then \( \mathcal{A} \) has a generator which is similar to a self-adjoint operator.

**Proof.** Write \( \mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \), where \( \mathcal{A}_1 \) has no type II finite summand and \( \mathcal{A}_2 \) is type II finite. By Theorem 1, \( \mathcal{A}_1 \) has a generator \( A_1 \) which is similar to a self-adjoint operator. Construct a generator of \( \mathcal{A}_2 \) as follows: Choose a projection \( E \in \mathcal{A}_2 \) such that \( \mathcal{A}_2 \) is spatially \(*\)-isomorphic to \( M(E, \mathcal{A}_2) \). \( E, \mathcal{A}_2 \) is type II finite, so \( E, \mathcal{A}_2 \) has a single generator by assumption. Now use Lemma 1 to construct a generator \( A_2 \) of \( \mathcal{A}_2 \) which is similar to a self-adjoint and such that \( \sigma(A_1) \) and \( \sigma(A_2) \) are disjoint. Then \( A_1 \oplus A_2 \) is similar to a self-adjoint operator, and \( \mathcal{B}(A_1 \oplus A_2) = \mathcal{A}_1 \oplus \mathcal{A}_2 \).

We now indicate briefly how the previous results can be obtained with “similar to a self-adjoint” replaced by “similar to a unitary,”

**Corollary 1’.** If \( \mathcal{A} \) is a properly infinite von Neumann algebra on \( \mathcal{H} \), then \( \mathcal{A} \) has a generator which is similar to a unitary operator.

The proof is the proof of Corollary 1, except that \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) must be chosen on the unit circle. (See [1] for another proof.)

**Theorem 1’.** If \( \mathcal{A} \) is a von Neumann algebra on a separable Hilbert space such that \( \mathcal{A} \) has no type II finite summand, then \( \mathcal{A} \) has a generator which is similar to a unitary operator.

**Proof.** Proceed as in the proof of Theorem 1. Write \( \mathcal{A} = \sum_{a=0}^\infty \oplus \mathcal{A}_a \). Use Lemmas 1 and 2 and Corollary 1’ to construct generators \( A_a \) of the \( \mathcal{A}_a \) which have pairwise disjoint spectra on the unit circle. Then each \( A_a \) will be similar to a unitary operator. To handle the summands \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), we need the following: If \( C \) is a self-adjoint generator of \( \mathcal{C} \), then \( e^{i\lambda} \) is a unitary generator of \( \mathcal{C} \) and \( \sigma(e^{i\lambda}) = \{ e^{i\lambda} : \lambda \in \sigma(C) \} \). The rest of the proof is clear.

Finally we have

**Theorem 2’.** If (S) is true and \( \mathcal{A} \) is a von Neumann algebra on a separable Hilbert space, then \( \mathcal{A} \) has a generator which is similar to a unitary operator.
REFERENCES


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