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**VON NEUMANN ALGEBRAS GENERATED BY OPERATORS  
SIMILAR TO NORMAL OPERATORS**

WARREN R. WOGEN

## VON NEUMANN ALGEBRAS GENERATED BY OPERATORS SIMILAR TO NORMAL OPERATORS

W. R. WOGEN

A normal operator generates an abelian von Neumann algebra. However, an operator which is similar to a normal operator may generate a von Neumann algebra which is not even type I. In fact, it is shown that if  $\mathcal{A}$  is a von Neumann algebra on a separable Hilbert space and  $\mathcal{A}$  has no type II finite summand, then  $\mathcal{A}$  has a generator which is similar to a self-adjoint and  $\mathcal{A}$  has a generator which is similar to a unitary. The restriction that  $\mathcal{A}$  have no type II finite summand can be removed provided that it is assumed that every type II finite von Neumann algebra has a single generator.

Let  $\mathcal{H}$  be a separable Hilbert space and let  $\mathcal{A}$  be a von Neumann algebra on  $\mathcal{H}$ .  $\mathcal{A}'$  denotes the commutant of  $\mathcal{A}$ . For  $n \geq 2$ , let  $M_n(\mathcal{A})$  denote the von Neumann algebra of  $n \times n$  matrices with entries in  $\mathcal{A}$ . If  $T$  is a bounded operator, the  $\mathcal{R}(T)$  is the von Neumann algebra generated by  $T$ .

We begin with some lemmas.

**LEMMA 1.** *Let  $\mathcal{A} = \mathcal{R}(C)$  and suppose  $n \geq 3$ . Let  $\{\lambda_k\}_{k=1}^n$  and  $\{a_k\}_{k=1}^{n-1}$  be sequences of complex numbers such that the  $\lambda_k$  are distinct, each  $a_k \neq 0$ , and  $\|(\lambda_1 - \lambda_2)C\| \leq |a_1 a_2|$ . Define  $A = (A_{i,j})_{i,j=1}^n \in M_n(\mathcal{A})$  by  $A_{k,k} = \lambda_k I$ ,  $A_{k+1,k} = a_k I$ ,  $A_{3,1} = C$ , and  $A_{i,j} = 0$  otherwise. Define  $B = (B_{i,j})_{i,j=1}^n \in M_n(\mathcal{A})$  by  $B_{k,k} = \lambda_k I$  and  $B_{i,j} = 0$  if  $i \neq j$ . Then  $A$  and  $B$  are similar, and  $\mathcal{R}(A) = M_n(\mathcal{A})$ .*

*Proof.* It follows from [11, Lemma 1] that  $\mathcal{R}(A) = M_n(\mathcal{A})$ . To show that  $A$  and  $B$  are similar we need only that the  $\lambda_k$  are distinct. We must find an invertible operator  $S$  such that  $AS = SB$ . Such an  $S$  of the form  $S = I + N$ , where  $N$  is lower triangular and nilpotent, can be computed easily. Merely perform the matrix multiplications and solve for the entries of  $S$ . We omit the details.

**REMARK 1.** If the operator  $S = I + N$  in Lemma 1 is computed, we see that we can make the entries of  $N$  small by choosing  $\|C\|$ ,  $|a_1|$ ,  $|a_2|$ ,  $\dots$ ,  $|a_{n-1}|$  suitably small. Hence we can suppose that  $\|N\| < 1/2$ . Then  $\|S\| = \|I + N\| < 3/2$  and  $\|S^{-1}\| = \|I - N + N^2 - \dots \pm N^{n-1}\| < 2$ . Note also that by choosing  $\|C\|$ ,  $|a_1|$ ,  $|a_2|$ ,  $\dots$ ,  $|a_{n-1}|$  suitably, we can assume that  $\|A\| \leq \|B\| + 1$ .

The following is a corollary of Lemma 1.

**COROLLARY 1.** *If  $\mathcal{A}$  is a properly infinite von Neumann algebra on  $\mathcal{H}$ , then  $\mathcal{A}$  has a generator which is similar to a self-adjoint operator.*

*Proof.* If  $\mathcal{A}$  is properly infinite, then it is well-known that  $\mathcal{A}$  is \*-isomorphic to  $M_3(\mathcal{A})$ .  $\mathcal{A}$  has a single generator  $C$  by [10]. Construct a generator  $A$  of  $M_3(\mathcal{A})$  as in Lemma 1, with  $\lambda_1, \lambda_2$ , and  $\lambda_3$  real. Then  $A$  is similar to self-adjoint operator by Lemma 1. (Another easy proof of Corollary 1 can be deduced from methods in the proof of Corollary 1 in [1].)

It has been shown that if  $\mathcal{A}$  is properly infinite, then  $\mathcal{A}$  is generated by three projections [9] and by two idempotents [4]. A related result is

**COROLLARY 2.** *If  $\mathcal{A}$  is a properly infinite von Neumann algebra on  $\mathcal{H}$ , then  $\mathcal{A}$  is generated by three commuting idempotents.*

*Proof.* If  $A$  is the generator of  $\mathcal{A}$  constructed in Corollary 1, let  $E$  be the (idempotent valued) spectral measure of  $A$ . Then  $E(\lambda_1)$ ,  $E(\lambda_2)$ , and  $E(\lambda_3)$  are the required commuting idempotents.

Let  $\sigma(C)$  denote the spectrum of the operator  $C$ .

**LEMMA 2.** *Let  $\mathcal{A} = \mathcal{R}(C)$ . Let*

$$A = \begin{bmatrix} C & 0 \\ aI & \lambda I \end{bmatrix}, \quad B = \begin{bmatrix} C & 0 \\ 0 & \lambda I \end{bmatrix},$$

where  $a \neq 0$  and  $\lambda \notin \sigma(C)$ . Then  $A$  is similar to  $B$ , and  $\mathcal{R}(A) = M_3(\mathcal{A})$ .

*Proof.* A routine computation shows that

$$\mathcal{R}(A)' = \left\{ \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} : T \in \mathcal{A}' \right\}.$$

It follows that  $\mathcal{R}(A) = \mathcal{R}(A)'' = M_3(\mathcal{A})$ . Let

$$S = \begin{bmatrix} I & 0 \\ a(C - \lambda I)^{-1} & I \end{bmatrix}.$$

Then  $S$  is invertible and  $AS = SB$ .

**LEMMA 3.** *Let  $\{A_k\}_{k=0}^{\infty}$  be a uniformly bounded sequence of operators. Suppose that the  $A_k$  have pairwise disjoint spectra. Then*

$$\mathcal{R}\left(\sum_{k=0}^{\infty} \oplus A_k\right) = \sum_{k=0}^{\infty} \oplus \mathcal{R}(A_k).$$

*Proof.* The proof given here is due essentially to Rosenthal [8, Th. 3]. (See also [3, Lemma].) Let  $A = \sum_{k=0}^{\infty} \oplus A_k$ . Suppose  $C = (C_{ij})_{i,j=0}^{\infty}$  commutes with  $A$ . Then

$$C_{i,j}A_j = A_iC_{i,j} \quad \text{for all } i, j.$$

If  $i \neq j$ , then  $\sigma(A_i)$  and  $\sigma(A_j)$  are disjoint, so by a theorem of Rosenblum [7],  $C_{ij} = 0$ . It follows that  $\mathcal{R}(A)' = \sum_{k=0}^{\infty} \oplus \mathcal{R}(A_k)'$ , so that  $\mathcal{R}(A) = \mathcal{R}(A)'' = \sum_{k=1}^{\infty} \oplus \mathcal{R}(A_k)$ .

**THEOREM 1.** *If  $\mathcal{A}$  is a von Neumann algebra on a separable Hilbert space such that  $\mathcal{A}$  has no type II finite summand, then  $\mathcal{A}$  has a generator which is similar to a self-adjoint operator.*

*Proof.* Write  $\mathcal{A} = \sum_{n=0}^{\infty} \oplus \mathcal{A}_n$ , where  $\mathcal{A}_0$  is properly infinite and for each  $n \geq 1$ ,  $\mathcal{A}_n$  is an  $n$ -homogeneous type I summand (see [2]). (Note that some of these summands may be absent.) Let  $\{I_n\}_{n=0}^{\infty}$  be a pairwise disjoint sequence of nonempty subintervals of  $[0, 1]$ .

By Corollary 1, we can choose  $A_0$  and an invertible operator  $S_0$  such that  $\mathcal{R}(A_0) = \mathcal{A}_0$ ,  $S_0A_0S_0^{-1}$  is self-adjoint, and  $\sigma(A_0) \subset I_0$ .

For each  $n \geq 1$ ,  $\mathcal{A}_n$  is  $*$ -isomorphic to  $M_n(\mathcal{C}_n)$ , where  $\mathcal{C}_n$  is the center of  $\mathcal{A}_n$  (see [2]).  $\mathcal{C}_n$  is abelian, so  $\mathcal{C}_n$  has a self-adjoint generator by [5]. Let  $A_1$  be a self-adjoint generator of  $\mathcal{A}_1 = \mathcal{C}_1$ . By translating and scaling, if necessary, we can assume  $\sigma(A_1) \subset I_1$ . Let  $S_1$  be the identity in  $\mathcal{A}_1$ .

Let  $C$  be a self-adjoint generator of  $\mathcal{C}_2$  with  $\sigma(C)$  properly contained in  $I_2$ . Let  $\lambda \in I_2$  with  $\lambda \notin \sigma(C)$ . Let  $a \neq 0$  and let

$$A_2 = \begin{bmatrix} C & 0 \\ aI & \lambda I \end{bmatrix}.$$

Then by Lemma 2,  $\mathcal{R}(A_2) = \mathcal{A}_2$  and for some invertible  $S_2$ ,  $S_2A_2S_2^{-1}$  is self-adjoint. Also,  $\sigma(A_2) = \sigma(C) \cup \{\lambda\} \subset I_2$ .

For  $n \geq 3$ , use Lemma 1 to construct  $A_n$  and an invertible  $S_n$  such that  $\mathcal{R}(A_n) = \mathcal{A}_n$ ,  $S_nA_nS_n^{-1}$  is self-adjoint, and  $\sigma(A_n) \subset I_n$ . Moreover by Remark 1, we can suppose that the sequences  $\{A_n\}$ ,  $\{S_n\}$ , and  $\{S_n^{-1}\}$  are uniformly bounded.

Let  $A = \sum_{n=0}^{\infty} \oplus A_n$ , and let  $S = \sum_{n=0}^{\infty} S_n$ . Then  $A$  and  $S$  are bounded operators,  $S$  is invertible, and  $SAS^{-1}$  is self-adjoint. Finally  $\mathcal{R}(A) = \sum_{n=0}^{\infty} \oplus A_n$  by Lemma 3.

It has long been conjectured that every von Neumann algebra on a separable Hilbert space has a single generator. Results in [6] and

[10] reduce the proof of the conjecture to showing that (S) Every type II finite von Neumann algebra on a separable Hilbert space has single generator. (See [4] for a partial solution to this conjecture.)

**THEOREM 2.** *If (S) is true and  $\mathcal{A}$  is a von Neumann algebra on a separable Hilbert space, then  $\mathcal{A}$  has a generator which is similar to a self-adjoint operator.*

*Proof.* Write  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ , where  $\mathcal{A}_1$  has no type II finite summand and  $\mathcal{A}_2$  is type II finite. By Theorem 1,  $\mathcal{A}_1$  has a generator  $A_1$  which is similar to a self-adjoint operator. Construct a generator of  $\mathcal{A}_2$  as follows: Choose a projection  $E \in \mathcal{A}_2$  such that  $\mathcal{A}_2$  is spatially \*-isomorphic to  $M_1(E\mathcal{A}_2E)$ .  $E\mathcal{A}_2E$  is type II finite, so  $E\mathcal{A}_2E$  has a single generator by assumption. Now use Lemma 1 to construct a generator  $A_2$  of  $\mathcal{A}_2$  which is similar to a self-adjoint and such that  $\sigma(A_1)$  and  $\sigma(A_2)$  are disjoint. Then  $A_1 \oplus A_2$  is similar to a self-adjoint operator, and  $\mathcal{P}(A_1 \oplus A_2) = \mathcal{A}_1 \oplus \mathcal{A}_2$ .

We now indicate briefly how the previous results can be obtained with "similar to a self-adjoint" replaced by "similar to a unitary,"

**COROLLARY 1'.** *If  $\mathcal{A}$  is a properly infinite von Neumann algebra on  $\mathcal{H}$ , then  $\mathcal{A}$  has a generator which is similar to a unitary operator.*

The proof is the proof of Corollary 1, except that  $\lambda_1, \lambda_2$ , and  $\lambda_3$  must be chosen on the unit circle. (See [1] for another proof.)

**THEOREM 1'.** *If  $\mathcal{A}$  is a von Neumann algebra on a separable Hilbert space such that  $\mathcal{A}$  has no type II finite summand, then  $\mathcal{A}$  has a generator which is similar to a unitary operator.*

*Proof.* Proceed as in the proof of Theorem 1. Write  $\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n$ . Use Lemmas 1 and 2 and Corollary 1' to construct generators  $A_n$  of the  $\mathcal{A}_n$  which have pairwise disjoint spectra on the unit circle. Then each  $A_n$  will be similar to a unitary operator. To handle the summands  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , we need the following: If  $C$  is a self-adjoint generator of  $\mathcal{C}$ , then  $e^{iC}$  is a unitary generator of  $\mathcal{C}$  and  $\sigma(e^{iC}) = \{e^{i\lambda} : \lambda \in \sigma(C)\}$ . The rest of the proof is clear.

Finally we have

**THEOREM 2'.** *If (S) is true and  $\mathcal{A}$  is a von Neumann algebra on a separable Hilbert space, then  $\mathcal{A}$  has a generator which is similar to a unitary operator.*

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