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**ON THE DEGREE OF THE MINIMAL POLYNOMIAL OF A
COMMUTATOR OPERATOR**

MOHAMMAD SHAFQAT ALI AND MARVIN DAVID MARCUS

ON THE DEGREE OF THE MINIMAL POLYNOMIAL OF A COMMUTATOR OPERATOR

M. SHAFQAT ALI AND MARVIN MARCUS

Let A be an n -square matrix over a field F of characteristic 0. The additive commutator operator defined by A , $T_A X = AX - XA$, can be regarded as a linear transformation on the space of all n -square matrices X over F . Following earlier papers by O. Taussky and H. Wielandt and one of the present authors, we show that the degree of the minimal polynomial of T_A is always odd and at least

$$2[m + E + (k - 2)e - k] + 1$$

where m is the degree of the minimal polynomial of A , k is the number of distinct eigenvalues of A , and $E(e)$ is the largest (least) integer among the degrees of the distinct highest degree elementary divisors of the characteristic matrix of A .

The purpose of this paper is two-fold: first we obtain a count of the number of distinct differences of the form $z_i - z_j$, $i \neq j$, where z_1, \dots, z_n are distinct elements of a field F of characteristic 0; second we apply this to prove a result on the parity and magnitude of the degree of the minimal polynomial of a matrix commutator. Annihilating polynomials for commutators were originally considered by Taussky and Wielandt in a paper in 1962 [5] and then again by one of the present authors in 1964 [2] (see also [1] and [6]).

To be precise let A be an n -square matrix over F and consider the linear transformation T_A defined on the space $M_n(F)$ of n -square matrices over F :

$$(1) \quad T_A X = AX - XA, \quad X \in M_n(F).$$

Then T_A is called the commutator operator defined by A . The transformation T_A has a matrix representation $A \otimes I_n - I_n \otimes A$ where \otimes indicates Kronecker product [3, p. 8]. The minimal polynomial of T_A is called the minimal polynomial of the commutator operator (1).

In an appropriate algebraic extension field K of F the elementary divisors of the characteristic matrix of A are powers of binomials. Let $\gamma_1, \dots, \gamma_k$ be the distinct eigenvalues of A , let e_i be the degree of the highest degree elementary divisor of the characteristic matrix of A involving γ_i , $i = 1, \dots, k$, let $E = \max_i e_i$, $e = \min_i e_i$, and let m be the degree of the minimal polynomial of A .

THEOREM 1. *If F is a field of characteristic zero then the degree*

of the minimal polynomial of the commutator operator T_A is always odd and at least

$$2[m + E + (k - 2)e - k] + 1.$$

In order to prove Theorem 1 we shall find it necessary to consider the following problem: given n distinct numbers z_1, \dots, z_n in F how many distinct differences are there of the form $z_i - z_j, i \neq j, i, j = 1, \dots, n$? Of course, the number can be as small as $2n - 2$ by simply taking $z_i = i, i = 1, \dots, n$. As an application of the Perron-Frobenius theorem on nonnegative matrices the following result, used to prove Theorem 1, may be of some independent interest.

THEOREM 2. *Let z_1, \dots, z_n be n distinct element in a field F of characteristic 0. Then there are always at least $2n - 2$ distinct non-zero differences of the form $z_i - z_j, i \neq j, i, j = 1, \dots, n$.*

II. *Proofs.* We begin with the proof of Theorem 2. We shall show in fact that there exists a permutation $\varphi \in S_n$ (the symmetric group of degree n) for which the $2n - 2$ elements

$$(2) \quad \pm(z_{\varphi(1)} - z_{\varphi(2)}), \dots, \pm(z_{\varphi(1)} - z_{\varphi(n)})$$

are distinct. If this is not the case then for every $\varphi \in S_n$ there must exist integers p and $q, p \neq q$, such that

$$(3) \quad z_{\varphi(1)} - z_{\varphi(p)} = z_{\varphi(q)} - z_{\varphi(1)}.$$

For, obviously the two sets of numbers (2) obtained by choosing first the + signs and then the - signs each consist of $n - 1$ distinct differences. Thus if there is to be an overlap, (3) must hold and we have $z_{\varphi(1)} = \frac{1}{2}z_{\varphi(p)} + \frac{1}{2}z_{\varphi(q)}$. Since φ is arbitrary we can write $z_i = \sum_{j=1}^n a_{ij}z_j, i = 1, \dots, n$, where for each i , there are precisely two values of j for which $a_{ij} = \frac{1}{2}$, and otherwise $a_{ij} = 0$. Let $A = (a_{ij}), z = (z_1, \dots, z_n)$ so that

$$(4) \quad Az = z.$$

The matrix A may or may not be reducible but in any event there exists an n -square permutation matrix P such that

$$(5) \quad P^T A P = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ * & A_2 & & \vdots \\ \vdots & & \cdot & \cdot & 0 \\ \vdots & & & & \cdot \\ * & \cdot & \cdot & \cdot & * & A_m \end{bmatrix}$$

and moreover each of the square matrices appearing along the main

diagonal in (5) is irreducible or 1-square. Now suppose A_1 is k -square. Since each row of A (and hence of P^TAP) has precisely two nonzero entries in it, it follows that $k \geq 2$. From (4) we have

$$(6) \quad P^TAPx = x$$

where $x = P^Tz$. Let $y = (x_1, \dots, x_k)$ and we see that (5) and (6) imply that

$$(7) \quad (A_1 - I_k)y = 0.$$

Since F has characteristic 0 it contains the rationals and A_1 can be regarded as a nonnegative, irreducible, row-stochastic matrix. By the Perron-Frobenius theorem [3, p. 124] we can immediately conclude that 1 is a simple eigenvalue of A_1 and hence the nullity of $A_1 - I_k$ over the rationals is $k - 1$. But the nullity is unchanged by regarding $A_1 - I_k$ as a matrix over any extension field of F . Now $e = (1, \dots, 1)$ is in the null space of $A_1 - I_k$ and hence any vector y satisfying (7) must be a multiple of e . Since $k \geq 2$ we conclude that at least two of the y_i are the same and hence that at least two of the z_i are the same. This contradiction completes the proof.

The preceding result has an immediate corollary. We let $\nu(\mathfrak{A})$ denote the cardinality of a set \mathfrak{A} .

COROLLARY. *Let \mathfrak{A} be the set of all distinct non-zero differences of the form $z_i - z_j, i \neq j$. Then $\nu(\mathfrak{A})$ is even and at least $2n - 2$.*

Proof. According to the preceding argument there exists a permutation $\varphi \in S_n$ such that the $2n - 2$ differences $\pm(z_{\varphi(i)} - z_{\varphi(i)}), i = 2, \dots, n$, are distinct. We can assume without loss of generality that φ is the identity. Let

$$\begin{aligned} \alpha &= \{z_1 - z_i, i = 2, \dots, n\}, \\ \beta &= \{z_i - z_1, i = 2, \dots, n\}, \end{aligned}$$

$\nu(\alpha) = \nu(\beta) = n - 1$. If $\mathfrak{A} = \alpha \cup \beta$ then we are finished. So assume that there exist integers $i, j, 1 < i \leq n, 1 < j \leq n, i \neq j$ such that $z_{i_1} - z_{j_1} \in \alpha \cup \beta$. But then clearly $z_{j_1} - z_{i_1} \in \alpha \cup \beta$. For if $z_{j_1} - z_{i_1} \in \alpha$, say, then $z_{j_1} - z_{i_1} = z_1 - z_i$ and hence $z_{i_1} - z_{j_1} = z_i - z_1$ in contradiction to the assumption that $z_{i_1} - z_{j_1} \in \alpha \cup \beta$. Now set

$$\alpha_1 = \alpha \cup \{z_{i_1} - z_{j_1}\}, \quad \beta_1 = \beta \cup \{z_{j_1} - z_{i_1}\}.$$

Clearly $\nu(\alpha_1 \cup \beta_1) = \nu(\alpha \cup \beta) + 2$ and if $\alpha_1 \cup \beta_1 \neq \mathfrak{A}$ we can repeat the preceding argument with α_1 and β_1 replacing α and β to obtain α_2 and β_2 such that $\nu(\alpha_2 \cup \beta_2) = \nu(\alpha_1 \cup \beta_1) + 2 = \nu(\alpha \cup \beta) + 4 = (2n - 2) + 4$. This procedure can obviously be continued until \mathfrak{A} is exhausted.

To prove Theorem 1 we use a well known theorem of Roth [4]: if $(\lambda - \gamma_i)^p$ and $(\lambda - \gamma_j)^q$ are a pair of elementary divisors of the characteristic matrix of A then corresponding to these is a list of elementary divisors of the characteristic matrix of $A \otimes I_n - I_n \otimes A$:

$$(\lambda - (\gamma_i - \gamma_j))^t,$$

where $t \leq p + q - 1$. According to Theorem 2 there are at least $(2k - 2)$ distinct nonzero differences of the form $\pm(\gamma_{\varphi(i)} - \gamma_{\varphi(j)})$, $j = 2, \dots, k$, and it is simply a matter of notational convenience to assume that these $2k - 2$ differences are $\pm(\gamma_1 - \gamma_i)$, $i = 2, \dots, k$. The highest degree elementary divisor involving the zero eigenvalue of the characteristic matrix of $A \otimes I_n - I_n \otimes A$ is

$$(8) \quad \lambda^{2E-1}.$$

By the corollary, the set \mathfrak{A} of all nonzero distinct eigenvalues of T_A is of the form

$$\begin{aligned} \mathfrak{A} = \{ & \pm(\gamma_1 - \gamma_i), i = 2, \dots, k \} \\ & \cup \{ \pm(\gamma_{i_t} - \gamma_{j_t}), t = 1, \dots, p \}. \end{aligned}$$

Now suppose the highest degree elementary divisors of the characteristic matrix of $A \otimes I_n - I_n \otimes A$ involving the nonzero distinct eigenvalues of T_A are:

$$\begin{aligned} & (\lambda - (\gamma_1 - \gamma_i))^{e_{r_i} + e_{s_i} - 1}, (\lambda - (\gamma_i - \gamma_1))^{e_{r_i} + e_{s_i} - 1}, i = 2, \dots, k, \\ & (\lambda - (\gamma_{i_t} - \gamma_{j_t}))^{e_{m_t} + e_{q_t} - 1}, (\lambda - (\gamma_{j_t} - \gamma_{i_t}))^{e_{m_t} + e_{q_t} - 1}, t = 1, \dots, p. \end{aligned}$$

Thus the degree of the minimal polynomial of T_A is

$$(9) \quad d = 2E - 1 + 2 \sum_{i=2}^k (e_{r_i} + e_{s_i} - 1) + 2 \sum_{t=1}^p (e_{m_t} + e_{q_t} - 1),$$

an odd integer. Observe that

$$d \geq (2E - 1) + 2 \sum_{i=2}^k (e_1 + e_i - 1) + 2 \sum_{t=1}^p (e_{i_t} + e_{j_t} - 1),$$

and hence

$$\begin{aligned} d & \geq (2E - 1) + 2 \sum_{i=2}^k e_i + 2(k - 1)(e_1 - 1) \\ & = 2E - 1 + 2(m - e_1) + 2(k - 1)(e_1 - 1) \\ & \geq 2[m + E + (k - 2)e - k] + 1. \end{aligned}$$

This completes the proof of Theorem 1.

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