

# Pacific Journal of Mathematics

**A CLASS OF COUNTEREXAMPLES ON PERMANENTS**

J. CSIMA

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**A method is described to construct a strictly positive doubly stochastic matrix  $A$  of order  $3k$  such that  $\text{per}(xE - A)$  has at least  $k$  real zeros.**

Let  $A$  be an irreducible doubly stochastic matrix. de Oliveira conjectured [1] that  $\text{per}(xE - A)$  has no real zeros or exactly one real zero depending on the parity of the order of  $A$ . We prove that the number of real zeros can be arbitrarily large for matrices of sufficiently large order, even or odd. We denote by  $E$  the identity matrix, always assuming its order to be such that the formulae make sense.

**LEMMA.** *There exist an infinite sequence  $A_1, A_2, \dots$  of doubly stochastic matrices of order 3 and a strictly increasing sequence of real numbers  $x_1, x_2, \dots$  such that  $\text{per}(x_t E - A_i) < 0$ , for  $t \leq i$ ,  $\text{per}(x_t E - A_i) > 0$ , for  $t > i$ , all  $i$ .*

*Proof.* Let  $0 < d < 1$ ,

$$A_d = \begin{bmatrix} 0 & d & 1-d \\ 1-d & 0 & d \\ d & 1-d & 0 \end{bmatrix} \text{ and } P_d(x) = \text{per}(xE - A_d).$$

Then  $P_d(x) = x^3 + 3d(1-d)(x+1) - 1$ , and we have  $P_d(-1) = -2 < 0$ ,  $P_d(1) = 6d(1-d) > 0$  and  $P'_d(x) = 3x^2 + 3d(1-d) > 0$ . Hence  $P_d$  is strictly increasing and has precisely one real zero which lies in the interval  $(-1, 1)$ . To each infinite sequence  $\{d_i\}$  ( $0 < d_i < 1$ ) we associate the sequence  $\{y_i\}$  where  $y_i(\text{real})$  is defined by  $P_{d_i}(y_i) = 0$ . Since  $\lim_{d \rightarrow 1} P_d(x) = x^3 - 1$ , there exists a strictly increasing sequence  $d_1 < d_2 < \dots$  such that the associated sequence of the  $y_i$  is strictly increasing. Setting  $x_1 = -1$ ,  $x_{i+1} = (y_i + y_{i+1})/2$  and  $A_i = A_{d_i}$  our lemma follows.

**THEOREM.** *For arbitrary positive integer  $k$  there exists a strictly positive doubly stochastic matrix  $A$  of order  $3k$  such that  $\text{per}(xE - A)$  has at least  $k$  distinct real zeros.*

*Proof.* Let us consider a pair of sequences  $\{A_n\}$  and  $\{x_n\}$  of our lemma and let  $B_k$  be the direct sum of  $A_1, A_2, \dots, A_k$ . Then  $\text{sgn}[\text{per}(x_i E - B_k)] = (-1)^{k-i+1}$  for  $i \leq k$ . Let  $\varepsilon > 0$  and  $B_{k,\varepsilon} = (1 + 3k\varepsilon)^{-1}$

$[B_k + \varepsilon J]$  where  $J$  is a matrix of ones. Since  $\lim_{\varepsilon \rightarrow 0} B_{k,\varepsilon} = B_k$  there exists a positive  $\varepsilon_0$  such that

$$\operatorname{sgn}[\operatorname{per}(x_i E - B_{k,\varepsilon_0})] = \operatorname{sgn}[\operatorname{per}(x_i E - B_k)] = (-1)^{k-i+1}$$

for  $i = 1, 2, \dots, k + 1$ . Then  $A = B_{k,\varepsilon_0}$  satisfies the requirements of the theorem.

Strictly positive matrices being irreducible, the above proof provides a method for actually constructing counterexamples for de Oliveira's conjecture. Choosing  $\varepsilon_0$  sufficiently small, one can even guarantee that  $\operatorname{per}(xE - A)$  has precisely  $k$  real zeros.

#### REFERENCE

1. G. N. de Oliveira, *A conjecture and some problems on permanents*, Pacific J. **32** (1970), 495-499.

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