ANALYTIC SHEAVES ON KLEIN SURFACES

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Morphisms of Klein surfaces are discussed from the sheaf-theoretic standpoint, and the cohomology of an analytic sheaf on a Klein surface is computed.

Let $\mathfrak{X}$ be a Klein surface $[1], [2]$; that is, $\mathfrak{X}$ consists of an underlying space $X$, which is a surface with boundary, and a family of equivalent dianalytic atlases on $X$. If $(U_a, z_a)$ is such an atlas, then $z_a: U_a \to \mathbb{C}^+$ is a homeomorphism of the open set $U_a$ in $X$ onto an open subset of $\mathbb{C}^+ = \{z \in \mathbb{C} | \text{Im}(z) \geq 0\}$. The functions $z_a$ must thus be real on $U_a \cap \partial X$, and it is required that $z_a \circ z_b^{-1}$ be dianalytic, that is, either analytic or antianalytic on each component of $z_b(U_a \cap U_b)$.

In this paper we define the structure sheaf of $\mathfrak{X}$, show that the concept of morphism given in $[1], [2]$ coincides with the concept of a morphism of ringed spaces, and compute the cohomology of analytic sheaves on $\mathfrak{X}$. If $\mathcal{F}$ is an analytic sheaf on $X$, and $\tilde{\mathcal{F}}$ is the lift of $\mathcal{F}$ to the complex double $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$, then there is a natural isomorphism

$$H^n(\tilde{\mathfrak{X}}, \tilde{\mathcal{F}}) \cong C \otimes_r H^n(\mathfrak{X}, \mathcal{F}).$$

1. The structure sheaf $\mathcal{O}_\mathfrak{X}$. We define the structure sheaf $\mathcal{O}_\mathfrak{X} = \mathcal{O}$ on $\mathfrak{X}$ as follows. If $U$ is open in $X$, let $\mathcal{O}(U)$ be the ring of holomorphic functions on $U$ (in the sense of $[1], [2]$). If $U \supset U'$, then the inclusion map is a morphism of Klein surfaces and we have a natural map $\rho_{U'}^U: \mathcal{O}(U) \to \mathcal{O}(U')$ (this is not quite an ordinary restriction map since the elements of $\mathcal{O}(U)$ are not quite functions). In particular, if $(U_a, z_a)$ and $(U_b, z_b)$ are dianalytic charts on $\mathfrak{X}$, $U_a \supset U_b$, then

$$\mathcal{O}(U_a) \cong \{f: U_a \to \mathbb{C} | f(U_a \cap \partial X) \subset \mathbb{R} \} \quad \text{and} \quad f \circ z_a^{-1} \text{ analytic}$$

and

$$\rho_{U_a}^{U_b}(f) = \begin{cases} f & \text{if } z_a \circ z_b^{-1} \text{ is analytic} \\ f|_{U_b} & \text{if } z_a \circ z_b^{-1} \text{ is antianalytic} \end{cases}.$$ 

It is easily checked that this defines a sheaf of local $\mathcal{R}$-algebras on $\mathfrak{X}$.

Let $\mathfrak{X}, \mathfrak{Y}$ be Klein surfaces, $f: \mathfrak{Y} \to \mathfrak{X}$ a continuous map. Then $f$ is a morphism $[1]$ if $f(\partial \mathfrak{Y}) \subset \partial \mathfrak{X}$ and if for every point $p \in \mathfrak{Y}$ there
are dianalytic charts \((V, w)\) and \((U, z)\) at \(p\) and \(f(p)\), and an analytic function \(h\) on \(w(V)\), such that

\[
\begin{array}{ccc}
V & \xrightarrow{f \mid V} & U \\
\downarrow w & & \downarrow z \\
C^+ & \xrightarrow{h} & C & \xrightarrow{\phi} & C^+
\end{array}
\]

commutes (\(\phi\) is the folding map, \(\phi(a + bi) = a + |b|i\)).

Recall that a ringed space morphism \(\mathcal{X} \to \mathcal{Y}\) is a pair \((f, \theta)\) where \(f: Y \to X\) is continuous and \(\theta: \mathcal{O}_X \to f_* \mathcal{O}_Y\) is a morphism of sheaves of rings [4, p. 36]. Here \(f_* \mathcal{O}_Y\) is the direct image sheaf: \(f_* \mathcal{O}_Y(U) = \mathcal{O}_Y(f^{-1}(U))\).

**Theorem 1.** Let \(\mathcal{X}, \mathcal{Y}\) be Klein surfaces, and let \(f: Y \to X\) be a nonconstant continuous map. Then the following are equivalent:

(i) \(f\) is a morphism;

(ii) there exists a morphism \(\theta: \mathcal{O}_X \to f_* \mathcal{O}_Y\) of sheaves of \(R\)-algebras.

Under these conditions the morphism \(\theta\) is unique, so \(f\) can be made in a unique way into a morphism of ringed spaces.

**Proof.** (i) \(\Rightarrow\) (ii). Let \(U \supset U'\) be open in \(X\). From the commutative diagram:

\[
\begin{array}{ccc}
f^{-1}(U) & \xleftarrow{f^{-1}(U')} & \xrightarrow{f} \mathcal{O}_Y(U) \\
\downarrow f & & \downarrow f \\
U & \xleftarrow{U'} & U'
\end{array}
\]

of morphisms of Klein surfaces we deduce a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_X(U') \\
\downarrow & & \downarrow \\
\mathcal{O}_Y(f^{-1}(U)) & \longrightarrow & \mathcal{O}_Y(f^{-1}(U'))
\end{array}
\]

of morphisms of \(R\)-algebras, and this defines an \(R\)-algebra morphism \(\theta: \mathcal{O}_X \to f_* \mathcal{O}_Y\).

(ii) \(\Rightarrow\) (i). Let \(p \in Y\), and let \((V, w), (U, z)\) be dianalytic charts at \(p, f(p)\), with \(f(V) \subset U\). Let \(z^*\) be the image of \(z\) in \(\mathcal{O}_Y(V)\) under

\[
\begin{array}{ccc}
\mathcal{O}_X(U) & \xrightarrow{f^{-1}} & \mathcal{O}_Y(U) \\
\downarrow & & \downarrow \\
\mathcal{O}_X(f^{-1}(U)) & \xrightarrow{f^{-1}} & \mathcal{O}_Y(f^{-1}(U'))
\end{array}
\]

Set \(h = z^* \circ w^{-1}\). We claim \(f \mid V = z^{-1} \circ \phi \circ h \circ w\), i.e. that \(z \circ (f \mid V) = \phi \circ z^*\). It clearly suffices to show that \(z(f(p)) = \phi(z^*(p))\). If this does not hold, then
The complex double. Let $\tilde{\mathfrak{X}}$ be a Klein surface, $\pi: \tilde{\mathfrak{X}} \to \mathfrak{X}$ its complex double. Recall that if $(U_a, z_a)$ is a dianalytic atlas on $\mathfrak{X}$, then $(\tilde{U}_a, \tilde{z}_a)$ is a dianalytic atlas on $\tilde{\mathfrak{X}}$, where $\tilde{U}_a = \pi^{-1}(U_a) = U'_a \cup U''_a$, $U'_a \cap U''_a = \pi^{-1}(U_a \cap \partial \mathfrak{X})$, and $\pi$ maps $U'_a$ and $U''_a$ each homeomorphically onto $U_a$. The function $\tilde{z}_a$ is defined by

$$\tilde{z}_a(p) = \begin{cases} z_a(p) & p \in U'_a \\ \overline{z_a(p)} & p \in U''_a \end{cases}.$$ 

$U'_a$ is identified with $U'_p$ where $z_a \circ z_b^{-1}$ is analytic, and with $U''_p$ where $z_a \circ z_b^{-1}$ is anti-analytic. This construction yields the Riemann surface (without boundary) $\tilde{\mathfrak{X}}$ as a double cover of $\mathfrak{X}$, folded along $\partial \mathfrak{X}$.

If $U$ is open in $\mathfrak{X}$, let $\tilde{U} = \pi^{-1}(U)$. We denote the structure sheaf of $\tilde{\mathfrak{X}}$ by $\tilde{\mathcal{O}}$. 

The complex double.
PROPOSITION 3. There is a canonical isomorphism

\[ C \otimes_R \mathcal{O}(U) \cong \mathcal{F}(\bar{U}) \]

for every open set \( U \subset X \).

Proof. We may cover \( U \) by dianalytic charts \((U_a, z_a)\). It then suffices to verify \((\dagger)\) for \( U_a \), since \( \mathcal{O}(U) \) is the difference kernel of \( \Pi_a \mathcal{O}(\bar{U}_a) \cong \Pi_{a,b} \mathcal{O}(\bar{U}_a \cap \bar{U}_b) \) and \( C \otimes_R \) is exact.

Let \( \sigma \) be the canonical anti-involution of \( \tilde{x} \) which commutes with \( \pi \), and let \( \kappa \) denote complex conjugation. If we identify \( \mathcal{O}(U_a) \) with its image in \( \mathcal{F}(\bar{U}_a) \) then we see

\[ \mathcal{O}(U_a) = \{ g \in \mathcal{F}(\bar{U}_a) \mid g = \kappa g \sigma \} . \]

But any \( g \in \mathcal{O}(U_a) \) can be written as

\[ g = \frac{1}{2}(g + \kappa g \sigma) + \frac{1}{2}(g - \kappa g \sigma) \]

and hence the canonical map

\[ C \otimes_R \mathcal{O}(U_a) \to \mathcal{F}(\bar{U}_a) \]

is surjective. This map is easily seen to be injective, completing the proof.

If \( \mathcal{F} \) is an analytic sheaf on \( \mathcal{X} \), let \( \tilde{\mathcal{F}} = \pi^* \mathcal{F} \).

THEOREM 4. There is a canonical isomorphism

\[ C \otimes_R \mathcal{F}(\mathcal{X}) \cong \tilde{\mathcal{F}}(\tilde{x}) . \]

Proof. We may choose a base for the topology of \( X \) consisting of sets of the form \( U_a \), where \( (U_a, z_a) \) is a dianalytic atlas on \( X \). Then sets of the form \( U'_a, U''_a \) (where \( U_a \cap \partial X = \emptyset \)) and of the form \( \bar{U}_a \) (where \( U_a \cap \partial X = \emptyset \)) form a base \( B \) for the topology of \( \tilde{x} \). Since \( \mathcal{F}(\bar{U}) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \cong C \otimes_R \mathcal{F}(U) \), it suffices to show that the sequence

\[ 0 \to \mathcal{F}(\tilde{\mathcal{X}}) \otimes_{\mathcal{F}(\mathcal{X})} \mathcal{F}(\mathcal{X}) \to \prod_{V \in B} \mathcal{F}(V) \otimes_{\mathcal{O}(\pi V)} \mathcal{F}(\pi V) \]

\[ \cong \prod_{V, W \in B} \mathcal{F}(V \cap W) \otimes_{\mathcal{O}(\pi (V \cap W))} \mathcal{F}(\pi(V \cap W)) \]

is exact. When \( U'_a \) and \( U''_a \) are disjoint then \( \mathcal{F}(\bar{U}_a) = \mathcal{F}(U'_a) \times \mathcal{F}(U''_a) \) so \((\dagger)\) may be replaced by
and this last is exact because of Proposition 3 and the fact that $\mathcal{F}$ is a sheaf.

Since the functors $\mathcal{F} \to C \otimes_R \mathcal{F}(x)$ and $\mathcal{F} \to \mathcal{F}(x)$ are canonically isomorphic, so are their derived functors [3], and we have

**Theorem 5.** Let $\mathcal{F}$ be an analytic sheaf on the Klein surface $x$. Then there is a canonical isomorphism

$$H^q(x, \mathcal{F}) \cong C \otimes_R H^q(x, \mathcal{F})$$

for all $q \geq 0$.

**Corollary.** (Cartan Theorem B) Let $x$ be a non-compact Klein surface, $\mathcal{F}$ a coherent analytic sheaf on $x$. Then $H^q(x, \mathcal{F}) = 0$ for all $q \geq 1$.

**Proof.** Use Theorem 5 and Proposition 2 to reduce to the case of a non-compact Riemann surface [6, p. 270].

**References**


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