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ANALYTIC SHEAVES ON KLEIN SURFACES

NEWCOMB GREENLEAF

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Morphisms of Klein surfaces are discussed from the sheaf-theoretic standpoint, and the cohomology of an analytic sheaf on a Klein surface is computed.

0. Let \mathfrak{X} be a Klein surface [1], [2]; that is, \mathfrak{X} consists of an underlying space X , which is a surface with boundary, and a family of equivalent dianalytic atlases on X . If (U_α, z_α) is such an atlas, then $z_\alpha: U_\alpha \rightarrow \mathbb{C}^+$ is a homeomorphism of the open set U_α in X onto an open subset of $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\}$. The functions z_α must thus be real on $U_\alpha \cap \partial X$, and it is required that $z_\alpha \circ z_\beta^{-1}$ be dianalytic, that is, either analytic or antianalytic on each component of $z_\beta(U_\alpha \cap U_\beta)$.

In this paper we define the structure sheaf of \mathfrak{X} , show that the concept of morphism given in [1], [2] coincides with the concept of a morphism of ringed spaces, and compute the cohomology of analytic sheaves on \mathfrak{X} . If \mathcal{F} is an analytic sheaf on X , and $\tilde{\mathcal{F}}$ is the lift of \mathcal{F} to the complex double $\tilde{\mathfrak{X}}$ of \mathfrak{X} , then there is a natural isomorphism

$$H^q(\tilde{\mathfrak{X}}, \tilde{\mathcal{F}}) \cong \mathbb{C} \otimes_{\mathbb{R}} H^q(\mathfrak{X}, \mathcal{F}).$$

1. The structure sheaf $\mathcal{O}_{\mathfrak{X}}$. We define the structure sheaf $\mathcal{O}_{\mathfrak{X}} = \mathcal{O}$ on \mathfrak{X} as follows. If U is open in X , let $\mathcal{O}(U)$ be the ring of holomorphic functions on U (in the sense of [1], [2]). If $U \supset U'$, then the inclusion map is a morphism of Klein surfaces and we have a natural map $\rho_{U'}^U: \mathcal{O}(U) \rightarrow \mathcal{O}(U')$ (this is not quite an ordinary restriction map since the elements of $\mathcal{O}(U)$ are not quite functions). In particular, if (U_α, z_α) and (U_β, z_β) are dianalytic charts on \mathfrak{X} , $U_\alpha \supset U_\beta$, then

$$\mathcal{O}(U_\alpha) \cong \left\{ \begin{array}{l} f: U_\alpha \rightarrow \mathbb{C} \mid f(U_\alpha \cap \partial X) \subset \mathbb{R}, \\ \text{and } f \circ z_\alpha^{-1} \text{ analytic} \end{array} \right\}$$

and

$$\rho_{U_\beta}^{U_\alpha}(f) = \begin{cases} f|_{U_\beta} \text{ where } z_\alpha \circ z_\beta^{-1} \text{ is analytic} \\ \bar{f}|_{U_\beta} \text{ where } z_\alpha \circ z_\beta^{-1} \text{ is antianalytic.} \end{cases}$$

It is easily checked that this defines a sheaf of local \mathbb{R} -algebras on \mathfrak{X} .

Let $\mathfrak{X}, \mathfrak{Y}$ be Klein surfaces, $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ a continuous map. Then f is a morphism [1] if $f(\partial Y) \subset \partial X$ and if for every point $p \in Y$ there

are dianalytic charts (V, w) and (U, z) at p and $f(p)$, and an analytic function h on $w(V)$, such that

$$\begin{array}{ccc} V & \xrightarrow{f|V} & U \\ w \downarrow & & \downarrow z \\ \mathbb{C}^+ & \xrightarrow{h} \mathbb{C} \xrightarrow{\phi} & \mathbb{C}^+ \end{array}$$

commutes (ϕ is the folding map, $\phi(a + bi) = a + |b|i$).

Recall that a ringed space morphism $\mathfrak{Y} \rightarrow \mathfrak{X}$ is a pair (f, θ) where $f: Y \rightarrow X$ is continuous and $\theta: \mathcal{O}_{\mathfrak{X}} \rightarrow f_* \mathcal{O}_{\mathfrak{Y}}$ is a morphism of sheaves of rings [4, p. 36]. Here $f_* \mathcal{O}_{\mathfrak{Y}}$ is the direct image sheaf: $f_* \mathcal{O}_{\mathfrak{Y}}(U) = \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U))$.

THEOREM 1. *Let $\mathfrak{X}, \mathfrak{Y}$ be Klein surfaces, and let $f: Y \rightarrow X$ be a nonconstant continuous map. Then the following are equivalent:*

- (i) *f is a morphism;*
- (ii) *there exists a morphism $\theta: \mathcal{O}_{\mathfrak{X}} \rightarrow f_* \mathcal{O}_{\mathfrak{Y}}$ of sheaves of \mathbf{R} -algebras.*

Under these conditions the morphism θ is unique, so f can be made in a unique way into a morphism of ringed spaces.

Proof. (i) \Rightarrow (ii). Let $U \supset U'$ be open in X . From the commutative diagram:

$$\begin{array}{ccc} f^{-1}(U) & \longleftarrow & f^{-1}(U') \\ f \downarrow & & \downarrow f \\ U & \longleftarrow & U' \end{array}$$

of morphisms of Klein surfaces we deduce a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}}(U) & \longrightarrow & \mathcal{O}_{\mathfrak{X}}(U') \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U)) & \longrightarrow & \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U')) \end{array}$$

of morphisms of \mathbf{R} -algebras, and this defines an \mathbf{R} -algebra morphism $\theta: \mathcal{O}_{\mathfrak{X}} \rightarrow f_* \mathcal{O}_{\mathfrak{Y}}$.

(ii) \Rightarrow (i). Let $p \in Y$, and let $(V, w), (U, z)$ be dianalytic charts at $p, f(p)$, with $f(V) \subset U$. Let z^* be the image of z in $\mathcal{O}_{\mathfrak{Y}}(V)$ under

$$(*) \quad \mathcal{O}_{\mathfrak{X}}(U) \rightarrow \mathcal{O}_{\mathfrak{Y}}(f^{-1}(U)) \rightarrow \mathcal{O}_{\mathfrak{Y}}(V) .$$

Set $h = z^* \circ w^{-1}$. We claim $f|V = z^{-1} \circ \phi \circ h \circ w$, i.e. that $z \circ (f|V) = \phi \circ z^*$. It clearly suffices to show that $z(f(p)) = \phi(z^*(p))$. If this does not hold, then

$$g = \frac{1}{[z - z^*(p)][\overline{z - z^*(p)}]}$$

is holomorphic at $f(p)$, and shrinking U, V if necessary, we may assume $g \in \mathcal{O}_{\mathfrak{x}}(U)$. We let g^* denote its image under $(*)$ in $\mathcal{O}_{\mathfrak{y}}(V)$. But $g^* = 1/[z^* - z^*(p)][\overline{z^* - z^*(p)}]$ which is not defined at p .

We still need to show that $f(\partial Y) \subset \partial X$. Let $q \in X$. Then $\mathcal{O}_{\mathfrak{x},q}$ is an \mathbf{R} -algebra which contains a copy of \mathbf{C} if and only if $q \notin \partial X$. The $\mathcal{O}_{\mathfrak{x},q}$ algebra $(f_*\mathcal{O}_{\mathfrak{x}})_q$ is isomorphic to

$$\prod_{f(p)=q} \mathcal{O}_{\mathfrak{y},p},$$

so $q \notin \partial X, f(p) = q$ implies $p \in \partial Y$.

We now check that θ is unique. Let U be open in $X, g \in \mathcal{O}_{\mathfrak{x}}(U), p \in f^{-1}(U)$. Let (V, w) be a dianalytic chart at p with $V \subset f^{-1}(U)$. Let g^* be the image of g in $\mathcal{O}_{\mathfrak{y}}(V)$ under $(*)$. Then using the above arguments, either $g^*(p) = gf(p)$ or $g^*(p) = \overline{gf(p)}$. If g is nonconstant, only one of these can yield an analytic function. If g is constant it can be expressed as a sum of nonconstant functions. Hence g^* , and thus θ , are uniquely determined. The theorem is proved.

By an analytic sheaf of \mathfrak{x} we mean an $\mathcal{O}_{\mathfrak{x}}$ -module. If \mathcal{F} is an analytic sheaf on \mathfrak{x} and $f: \mathfrak{y} \rightarrow \mathfrak{x}$ is a morphism then $f^*\mathcal{F}$ is the sheaf associated to the presheaf $V \rightarrow \mathcal{O}_{\mathfrak{y}}(V) \otimes_{\mathcal{O}_{\mathfrak{x}}(fV)} \mathcal{F}(fV)$.

PROPOSITION 2. *If \mathcal{F} is a coherent analytic sheaf on \mathfrak{x} , then $f^*\mathcal{F}$ is a coherent analytic sheaf on \mathfrak{y} .*

Proof. The proof given in [5, p. 47] for Riemann surfaces carries over to the Klein surface case.

2. The complex double. Let \mathfrak{x} be a Klein surface, $\pi: \tilde{\mathfrak{x}} \rightarrow \mathfrak{x}$ its complex double. Recall that if (U_α, z_α) is a dianalytic atlas on \mathfrak{x} , then $(\tilde{U}_\alpha, \tilde{z}_\alpha)$ is a dianalytic atlas on $\tilde{\mathfrak{x}}$, where $\tilde{U}_\alpha = \pi^{-1}(U_\alpha) = U'_\alpha \cup U''_\alpha, U'_\alpha \cap U''_\alpha = \pi^{-1}(U_\alpha \cap \partial X)$, and π maps U'_α and U''_α each homeomorphically onto U_α . The function \tilde{z}_α is defined by

$$\tilde{z}_\alpha(p) = \begin{cases} z_\alpha(p) & p \in U'_\alpha \\ \overline{z_\alpha(p)} & p \in U''_\alpha \end{cases}.$$

U'_α is identified with U'_β where $z_\alpha \circ z_\beta^{-1}$ is analytic, and with U''_β where $z_\alpha \circ \overline{z_\beta^{-1}}$ is anti-analytic. This construction yields the Riemann surface (without boundary) $\tilde{\mathfrak{x}}$ as a double cover of \mathfrak{x} , folded along ∂X .

If U is open in X , let $\tilde{U} = \pi^{-1}(U)$. We denote the structure sheaf of $\tilde{\mathfrak{x}}$ by $\tilde{\mathcal{O}}$.

PROPOSITION 3. *There is a canonical isomorphism*

$$(\dagger) \quad C \otimes_R \mathcal{O}(U) \cong \tilde{\mathcal{O}}(\tilde{U})$$

for every open set $U \subset X$.

Proof. We may cover U by dianalytic charts (U_α, z_α) . It then suffices to verify (\dagger) for U_α , since $\mathcal{O}(U)$ is the difference kernel of $\prod_\alpha \tilde{\mathcal{O}}(\tilde{U}_\alpha) \rightrightarrows \prod_{\alpha, \beta} \tilde{\mathcal{O}}(\tilde{U}_\alpha \cap \tilde{U}_\beta)$ and $C \otimes_R$ is exact.

Let σ be the canonical anti-involution of $\tilde{\mathfrak{X}}$ which commutes with π , and let κ denote complex conjugation. If we identify $\mathcal{O}(U_\alpha)$ with its image in $\tilde{\mathcal{O}}(\tilde{U}_\alpha)$ then we see

$$\mathcal{O}(U_\alpha) = \{g \in \tilde{\mathcal{O}}(\tilde{U}_\alpha) \mid g = \kappa g \sigma\} .$$

But any $g \in \mathcal{O}(U_\alpha)$ can be written as

$$g = \frac{1}{2}(g + \kappa g \sigma) + \frac{1}{2}(g - \kappa g \sigma)$$

and hence the canonical map

$$C \otimes_R \mathcal{O}(U_\alpha) \rightarrow \tilde{\mathcal{O}}(\tilde{U}_\alpha)$$

is surjective. This map is easily seen to be injective, completing the proof.

If \mathcal{F} is an analytic sheaf on \mathfrak{X} , let $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$.

THEOREM 4. *There is a canonical isomorphism*

$$C \otimes_R \mathcal{F}(\mathfrak{X}) \cong \tilde{\mathcal{F}}(\tilde{\mathfrak{X}}) .$$

Proof. We may choose a base for the topology of X consisting of sets of the form U_α , where (U_α, z_α) is a dianalytic atlas on X . Then sets of the form U'_α, U''_α (where $U_\alpha \cap \partial X = \emptyset$) and of the form \tilde{U}_α (where $U_\alpha \cap \partial X \neq \emptyset$) form a base B for the topology of $\tilde{\mathfrak{X}}$. Since $\tilde{\mathcal{O}}(\tilde{U}) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \cong C \otimes_R \mathcal{F}(U)$, it suffices to show that the sequence

$$\begin{aligned}
 (\dagger\dagger) \quad 0 &\rightarrow \tilde{\mathcal{O}}(\tilde{\mathfrak{X}}) \otimes_{\mathcal{O}(\mathfrak{X})} \mathcal{F}(\mathfrak{X}) \rightarrow \prod_{V \in B} \tilde{\mathcal{O}}(V) \otimes_{\mathcal{O}(\pi V)} \mathcal{F}(\pi V) \\
 &\rightrightarrows \prod_{V, W \in B} \tilde{\mathcal{O}}(V \cap W) \otimes_{\mathcal{O}(\pi(V \cap W))} \mathcal{F}(\pi(V \cap W)) .
 \end{aligned}$$

is exact. When U'_α and U''_α are disjoint then $\tilde{\mathcal{O}}(\tilde{U}_\alpha) = \tilde{\mathcal{O}}(U'_\alpha) \times \tilde{\mathcal{O}}(U''_\alpha)$ so $(\dagger\dagger)$ may be replaced by

$$\begin{aligned}
0 &\rightarrow \tilde{\mathcal{O}}(\mathfrak{X}) \otimes_{\mathcal{O}(\mathfrak{X})} \mathcal{F}(\mathfrak{X}) \rightarrow \prod_{\alpha} \tilde{\mathcal{O}}(\tilde{U}_{\alpha}) \otimes_{\mathcal{O}(U_{\alpha})} \mathcal{F}(U_{\alpha}) \\
&\Rightarrow \prod_{\alpha, \beta} \tilde{\mathcal{O}}(\tilde{U}_{\alpha\beta}) \otimes_{\mathcal{O}(U_{\alpha\beta})} \mathcal{F}(U_{\alpha\beta})
\end{aligned}$$

and this last is exact because of Proposition 3 and the fact that \mathcal{F} is a sheaf.

Since the functors $\mathcal{F} \rightarrow C \otimes_R \mathcal{F}(\mathfrak{X})$ and $\mathcal{F} \rightarrow \tilde{\mathcal{F}}(\tilde{\mathfrak{X}})$ are canonically isomorphic, so are their derived functors [3], and we have

THEOREM 5. *Let \mathcal{F} be an analytic sheaf on the Klein surface \mathfrak{X} . Then there is a canonical isomorphism*

$$H^q(\tilde{\mathfrak{X}}, \tilde{\mathcal{F}}) \cong C \otimes_R H^q(\mathfrak{X}, \mathcal{F})$$

for all $q \geq 0$.

COROLLARY. (Cartan Theorem B) *Let \mathfrak{X} be a non-compact Klein surface, \mathcal{F} a coherent analytic sheaf on \mathfrak{X} . Then $H^q(\mathfrak{X}, \mathcal{F}) = 0$ for all $q \geq 1$*

Proof. Use Theorem 5 and Proposition 2 to reduce to the case of a non-compact Riemann surface [6, p. 270].

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Pacific Journal of Mathematics

Vol. 37, No. 3

March, 1971

Mohammad Shafqat Ali and Marvin David Marcus, <i>On the degree of the minimal polynomial of a commutator operator</i>	561
Howard Anton and William J. Pervin, <i>Integration on topological semifields</i>	567
Martin Bartelt, <i>Multipliers and operator algebras on bounded analytic functions</i>	575
Donald Earl Bennett, <i>Aposyndetic properties of unicoherent continua</i>	585
James W. Bond, <i>Lie algebras of genus one and genus two</i>	591
Mario Borelli, <i>The cohomology of divisorial varieties</i>	617
Carlos R. Borges, <i>How to recognize homeomorphisms and isometries</i>	625
J. C. Breckenridge, <i>Burkill-Cesari integrals of quasi additive interval functions</i>	635
J. Csima, <i>A class of counterexamples on permanents</i>	655
Carl Hanson Fitzgerald, <i>Conformal mappings onto ω-swirly domains</i>	657
Newcomb Greenleaf, <i>Analytic sheaves on Klein surfaces</i>	671
G. Goss and Giovanni Viglino, <i>C-compact and functionally compact spaces</i>	677
Charles Lemuel Hagopian, <i>Arcwise connectivity of semi-aposyndetic plane continua</i>	683
John Harris and Olga Higgins, <i>Prime generators with parabolic limits</i>	687
David Michael Henry, <i>Stratifiable spaces, semi-stratifiable spaces, and their relation through mappings</i>	697
Raymond D. Holmes, <i>On contractive semigroups of mappings</i>	701
Joseph Edmund Kist and P. H. Maserick, <i>BV-functions on semilattices</i>	711
Shūichirō Maeda, <i>On point-free parallelism and Wilcox lattices</i>	725
Gary L. Musser, <i>Linear semiprime $(p; q)$ radicals</i>	749
William Charles Nemitz and Thomas Paul Whaley, <i>Varieties of implicative semilattices</i>	759
Jaroslav Nešetřil, <i>A congruence theorem for asymmetric trees</i>	771
Robert Anthony Nowlan, <i>A study of H-spaces via left translations</i>	779
Gert Kjærgaard Pedersen, <i>Atomic and diffuse functionals on a C*-algebra</i>	795
Tilak Raj Prabhakar, <i>On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials</i>	801
Leland Edward Rogers, <i>Mutually aposyndetic products of chainable continua</i>	805
Frederick Stern, <i>An estimate for Wiener integrals connected with squared error in a Fourier series approximation</i>	813
Leonard Paul Sternbach, <i>On k-shrinking and k-boundedly complete basic sequences and quasi-reflexive spaces</i>	817
Pak-Ken Wong, <i>Modular annihilator A*-algebras</i>	825