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***C*-COMPACT AND FUNCTIONALLY COMPACT SPACES**

G. GOSS AND GIOVANNI VIGLINO

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In the first section of this note a question posed by G. Viglino is resolved by constructing a C -compact space which is not seminormal. In the second section some characterizations of C -compact and functionally compact spaces are introduced. In the final section, embedding theorems of spaces into C -compact and functionally compact spaces are noted.

1. A Counterexample.

DEFINITIONS. (a) A Hausdorff space X is *absolutely closed* if given an open cover \mathcal{S} of X , then there exists a finite number of elements of \mathcal{S} ; say V_i , $1 \leq i \leq n$, with $X \subset \text{Cl } \bigcup_{i=1}^n V_i$.

(b) A Hausdorff space (X, τ) is *C -compact* if given a closed set Q of X and a τ -open cover \mathcal{S} of Q , then there exists a finite number of elements of \mathcal{S} ; say V_i , $1 \leq i \leq n$, with $Q \subset \text{Cl}_X \bigcup_{i=1}^n V_i$.

(c) An open set V is *regular* if $V = \bar{V}^\circ$.

(d) A space X is *seminormal* if given a closed subset C of X and an open set V containing C , then there exists a regular open set R with $C \subset R \subset V$.

G. Viglino has shown that a seminormal absolutely closed space is C -compact, and posed the question as to whether or not the converse holds [5]. The following is an example of a C -compact space which is not seminormal. An example has also been obtained by T. Lominac, Abstract #682-54-33.

EXAMPLE. Let Z represent the set of positive integers. Let

$$X = \left\{ \left(\frac{1}{2n-1}, \frac{1}{m} \right) \mid n, m \in Z \right\} \cup \left\{ \left(\frac{1}{2n}, -\frac{1}{m} \right) \mid n, m \in Z \right\} \\ \cup \left\{ \left(\frac{1}{n}, 0 \right) \mid n \in Z \right\} \cup \{ \infty \}.$$

Topologize X as follows. Partition Z into infinitely many infinite equivalence classes, $\{Z_i\}_{i=1}^\infty$, and let $\{Z'_j\}_{j=1}^\infty$ be a partition of Z_1 into infinitely many infinite equivalent classes. Let Φ denote a bijection from $\{(1/(2n-1), 1/m) \mid n, m \in Z\}$ to $Z \setminus \{1\}$. Let a neighborhood system for the points of the form $(1/(2i-1), 0)$ be composed of all sets of the form $U_{(2i-1, 0; k)} = V \cup F$ where $V = \{(1/(2i-1), 0)\} \cup \{(1/(2i-1), 1/m) \mid m \geq k\}$ and $F = \{(1/(2n-1), 1/m) \mid m \in Z_i \text{ and } n \geq k\} \cup \{(1/2n, -1/s) \mid$

$s \in \bigcup_{m \in \mathbb{Z}_i} Z_{\Phi(1/(2n-1), 1/m)}$ and $h \geq k$ for some $k \in \mathbb{Z}$. Let a system for $(1/(2i-1), 1/J)$ be the sets

$$\bigcup_{(2i-1, J; k)} \left\{ \left(\frac{1}{2i-1}, \frac{1}{J} \right) \right\} \cup \left\{ \left(\frac{1}{2n}, -\frac{1}{s} \right) \mid s \in Z_{\Phi(1/(2i-1), 1/J)} \text{ and } n \geq k \right\}$$

for some $k \in \mathbb{Z}$. The points $(1/2n, -1/m)$ are to be open, and a neighborhood system for the points of the form $(1/2i, 0)$ consists of sets of the form $\bigcup_{(2i, 0; k)} = V \cup F$ where $V = \{(1/2i, 0)\} \cup \{(1/2i, -1/m) \mid m \geq k\}$ and $F = \{(1/2n, -1/m) \mid m \in \mathbb{Z}_i^+ \text{ and } n \geq k\}$ for some $k \in \mathbb{Z}$. Finally, let a system for the point ∞ be composed of those sets of the form $X \setminus T$, where $T = \{(1/n, 0) \mid n \in \mathbb{Z}\} \cup \bigcup_{i=1}^k (\bigcup_{(2i-1, 0; k)} \cup \bigcup_{(2i, 0; k)})$ for $k \in \mathbb{Z}$.

An argument similar to that given in Example 1 of [4] may be used to show that X is a C -compact topological space. To show X is not seminormal we use the following characterization: X is seminormal if and only if given any closed subset C of X and any closed subset D disjoint from C , then there exists an open set U with $C \subset Cl U$ and $Cl U \cap D = \emptyset$ [5].

Let $C = \{(1/(2n-1), 0) \mid n \in \mathbb{Z}\}$, $D = \{(1/2n, 0) \mid n \in \mathbb{Z}\} \cup \{\infty\}$, and U any open set with $C \subset Cl U$. If $U \not\subset \{(1/2n, -1/m) \mid m, n \in \mathbb{Z}\}$ then clearly $Cl U \cap D \neq \emptyset$. If infinitely many elements of U are contained in $\{(1/2i, -1/m) \mid m \in \mathbb{Z}\}$ for some i , then $(1/2i, 0) \in Cl U$. On the other hand, if $U \subset \{(1/2n, -1/m) \mid m, n \in \mathbb{Z}\}$ and $U \cap \{(1/2i, -1/m) \mid m \in \mathbb{Z}\}$ is finite for each $i \in \mathbb{Z}$, then $\infty \in Cl U$. Hence X is not C -compact.

2. Characterizations of C -compact and functionally compact spaces.

DEFINITIONS. (a) A Hausdorff space X is *functionally compact* if for every open filter \mathcal{U} in X such that the intersection A of the elements of \mathcal{U} equals the intersection of the closure of the elements of \mathcal{U} , then \mathcal{U} is the neighborhood filter of A .

(b) A closed subset C of a space X is *regular closed* if for any $x \in C$ there exists an open neighborhood U_x with $Cl U_x \cap C = \emptyset$.

(c) Let S be a subset of a space X . An open cover $\{U_\alpha\}_{\alpha \in A}$ of S will be said to be a *regular cover* if $X \setminus \bigcup_{\alpha \in A} U_\alpha$ is a regular closed set.

(d) A space X is *regular seminormal* if given a regular closed set C of X and an open set V containing C , then there exists a regular open set R with $C \subset R \subset V$.

THEOREM 1. *The following properties are equivalent.*

- (i) X is functionally compact.
- (ii) Every continuous function from X into any Hausdorff space is closed.

(iii) Given a regular closed subset C of X , an open cover \mathcal{B} of $X \setminus C$, and an open neighborhood U of C , then there exist $O_i \in \mathcal{B}$, $1 \leq i \leq n$, such that $X = U \cup Cl_X \bigcup_{i=1}^n O_i$.

(iv) Given an open regular cover \mathcal{B} of any closed set C , then there exist $O_i \in \mathcal{B}$, $1 \leq i \leq n$ such that $C \subset Cl_X \bigcup_{i=1}^n O_i$.

(v) X is absolutely closed and regular seminormal.

Proof. The equivalence of (i) and (ii) has been shown by Dickman and Zame [1]. The statement that (i) and (iii) are equivalent is in [3]. (iii) \Rightarrow (iv). Let $\mathcal{B} = \{O_\alpha\}_{\alpha \in A}$ be a regular open cover for the closed set C . Then $D = X \setminus \bigcup_{\alpha \in A} O_\alpha$ is regular closed, \mathcal{B} is a cover of $X \setminus D$, and $X \setminus C$ is an open neighborhood of D . Hence there exist $O_i \in \mathcal{B}$, $1 \leq i \leq n$, such that $X = (X \setminus C) \cup Cl_X \bigcup_{i=1}^n O_i$; that is, $C \subset Cl_X \bigcup_{i=1}^n O_i$. A similar argument shows (iv) \Rightarrow (iii).

(iii) \Rightarrow (v). That X is absolutely closed may be seen by choosing in (iii) the empty set for both C and U . We show X is regular seminormal. Consider a regular closed set C and open set V containing C . For each $x \in X \setminus C$ choose a neighborhood U_x with $Cl U_x \cap C = \emptyset$. By (iii) there exist O_{x_i} , $1 \leq i \leq n$, with $C \subset X \setminus Cl_X \bigcup_{i=1}^n O_{x_i} \subset V$, and clearly $X \setminus Cl_X \bigcup_{i=1}^n O_{x_i}$ is regular open.

(v) \Rightarrow (iii). Let an open cover \mathcal{B} of $X \setminus C$ be given where C is regular closed. Let U be a neighborhood of C . Choose a regular open set R and $O_i \in \mathcal{B}$, $1 \leq i \leq n$, such that $C \subset R \subset U$ and $X = Cl \bigcup_{i=1}^n O_i$. Since R is a regular open set we have $Cl R \setminus R \subset Cl \bigcup_{i=1}^n O_i$ so that $X = U \cup Cl \bigcup_{i=1}^n O_i$.

DEFINITION (e). A Hausdorff space (X, τ) is *rim C-compact* (*rim functionally compact*) if there exists a neighborhood system for each point of X consisting of open sets, V , with the property that given a closed set Q of $Cl V \setminus V$ and a τ -open cover (regular cover) \mathcal{V} of Q , then there exists $V_i \in \mathcal{V}$, $1 \leq i \leq n$, with $Q \subset Cl_X \bigcup_{i=1}^n V_i$.

THEOREM 2. A Hausdorff space X is *C-Compact* (functionally compact) if and only if it is absolutely closed and rim C-compact (rim functionally compact).

Proof. A C -compact space is clearly absolutely closed and rim C -compact. To prove the converse we use the following obvious characterization of C -compactness: X is C -compact if and only if given any open cover \mathcal{V} of X , and any $V \in \mathcal{V}$ then there exist $V_i \in \mathcal{V}$, $1 \leq i \leq n$, with $X \subset V \cup Cl \bigcup_{i=1}^n V_i$. Let then $V \in \mathcal{V}$ where \mathcal{V} is an open cover of X . Choose for each $x \in V$ a rim C -compact neighborhood V_x with $V_x \subset V$. Select from the cover $(\mathcal{V} \setminus \{V\}) \cup \{V_x | x \in V\}$ elements

$V_i \in \mathcal{V} \setminus \{V\}$, $1 \leq i \leq n$, and $V_{x_i} \in \{V_x | x \in V\}$, $1 \leq i \leq m$, with

$$X = Cl \left(\bigcup_{i=1}^n V_i \cup \bigcup_{i=1}^m V_{x_i} \right).$$

Let $V_J^{(i)} \in \mathcal{V} \setminus \{V\}$, $1 \leq J \leq n_i$, be such that $Cl V_{x_i} \setminus V \subset Cl \bigcup_{J=1}^{n_i} V_J^{(i)}$, $1 \leq i \leq m$. Then

$$X = V \cup Cl \left(\bigcup_{i=1}^n V_i \cup \bigcup_{\substack{1 \leq i \leq m \\ 1 \leq J \leq n_i}} V_J^{(i)} \right).$$

A functionally compact space is clearly absolutely closed and rim functionally compact. The converse may be proved by using Theorem 1 (iii) and an argument similar to the one given above.

3. **Embeddings.** The absolute closed extensions constructed in the following theorem are of the type described by Fomin [2].

THEOREM 3. *Any rim C-compact (rim functionally compact) space can be embedded as a dense subspace of a C-compact (functionally compact) space.*

Proof. Let X be rim C-compact (rim functionally compact). Let \mathcal{B} denote the base of all open sets whose boundaries are C-compact (functionally compact). Let $X = \{\xi_x | x \in X\}$ where $\xi_x = \{O \in \mathcal{B} | x \in O\}$ and let Y denote the set of all maximal \mathcal{B} -filters with empty adherent set. Topologize $E = X \cup Y$ as follows. Let a neighborhood system for an element $\xi \in E$ be composed of all sets of the form $O_\xi = \{\xi' \in E | O \in \xi'\}$ for $O \in \xi$. Since \mathcal{B} is a complemented base, that is $\bar{O} \in \mathcal{B}$ for $O \in \mathcal{B}$, we have that E is an absolutely closed extension of X [2]. To show that E is C-compact (functionally compact) it is sufficient, by §2 Theorem 2, to show that E is rim C-compact (rim functionally compact). To do this we note that for any open set O of X , $Cl_E O_E \setminus O_E = O_E \cup \{\xi_x | x \in Cl_X O\}$, so that $\{O_E | O \in \mathcal{B}\}$ is an open base for E with boundary of O_E C-compact (functionally compact).

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