C-COMPACT AND FUNCTIONALLY COMPACT SPACES

G. GOSS AND GIOVANNI VIGLINO
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In the first section of this note a question posed by G. Viglino is resolved by constructing a C-compact space which is not seminormal. In the second section some characterizations of C-compact and functionally compact spaces are introduced. In the final section, embedding theorems of spaces into C-compact and functionally compact spaces are noted.

1. A Counterexample.

DEFINITIONS. (a) A Hausdorff space $X$ is absolutely closed if given an open cover $\mathcal{V}$ of $X$, then there exists a finite number of elements of $\mathcal{V}$, say $V_i$, $1 \leq i \leq n$, with $X \subseteq \mathrm{Cl} \bigcup_{i=1}^{n} V_i$.

(b) A Hausdorff space $(X, \tau)$ is C-compact if given a closed set $Q$ of $X$ and a $\tau$-open cover $\mathcal{V}$ of $Q$, then there exists a finite number of elements of $\mathcal{V}$, say $V_i$, $1 \leq i \leq n$, with $Q \subseteq \mathrm{Cl}_X \bigcup_{i=1}^{n} V_i$.

(c) An open set $V$ is regular if $V = \overline{V}$.

(d) A space $X$ is seminormal if given a closed subset $C$ of $X$ and an open set $V$ containing $C$, then there exists a regular open set $R$ with $C \subseteq R \subseteq V$.

G. Viglino has shown that a seminormal absolutely closed space is C-compact, and posed the question as to whether or not the converse holds [5]. The following is an example of a C-compact space which is not seminormal. An example has also been obtained by T. Lominac, Abstract #682-54-33.

EXAMPLE. Let $Z$ represent the set of positive integers. Let

$$X = \left\{ \left( \frac{1}{2n-1}, \frac{1}{m} \right) \mid n, m \in \mathbb{Z} \right\} \cup \left\{ \left( \frac{1}{2n}, -\frac{1}{m} \right) \mid n, m \in \mathbb{Z} \right\}
\cup \left\{ \left( \frac{1}{n}, 0 \right) \mid n \in \mathbb{Z} \right\} \cup \{ \infty \}.$$

Topologize $X$ as follows. Partition $Z$ into infinitely many infinite equivalence classes, $\{Z_i\}_{i=1}^\infty$, and let $\{Z_i'\}_{i=1}^\infty$ be a partition of $Z_i$ into infinitely many infinite equivalent classes. Let $\Phi$ denote a bijection from $\{(1/(2n-1), 1/m) \mid n, m \in \mathbb{Z} \}$ to $Z \setminus \{1\}$. Let a neighborhood system for the points of the form $(1/(2i-1), 0)$ be composed of all sets of the form $U = \{1/(2i-1), 0\} \cup \{(1/(2i-1), 1/m) \mid m \geq k\}$ and $F = \{(1/(2n-1), 1/m) \mid m \in Z_i \mbox{ and } n \geq k\} \cup \{(1/2n, -1/s) \mid s \geq 1\}$.
s ∈ \bigcup_{m \in Z} Z_{\Phi(1/(2n-1), 1/m)} and n \geq k} for some k ∈ Z. Let a system for \((1/(2i - 1), 1/J)\) be the sets
\[
\bigcup_{(2i-1, J, k)} \left\{ \left( \frac{1}{2i - 1}, \frac{1}{J} \right) \right\} \cup \left\{ \left( \frac{1}{2n}, - \frac{1}{s} \right) \left| s \in Z_{\Phi(1/(2i-1), 1/J)} \text{ and } n \geq k \right\}
\]
for some k ∈ Z. The points \((1/2n, -1/m)\) are to be open, and a neighborhood system for the points of the form \((1/2i, 0)\) consists of sets of the form \(U \left( \frac{1}{2i}, 0 \right)\) where \(V = \{ (1/2n, 0) \} \cup \{(1/2i, -1/m) \mid m \geq k \}\) and \(F = \{ (1/2n, -1/m) \mid m \in Z \} \) and \(n \geq k \) for some \(k \in Z\). Finally, let a system for the point \(\infty\) be composed of those sets of the form \(X \setminus T\), where \(T = \{ (1/n, 0) \mid n \in Z \} \cup \bigcup_{i=1}^{k} (U_{i} \left( \frac{1}{2i-1}, 0, k \right) \cup U_{\infty})\) for \(k \in Z\).

An argument similar to that given in Example 1 of [4] may be used to show that \(X\) is a C-compact topological space. To show \(X\) is not seminormal we use the following characterization: \(X\) is seminormal if and only if given any closed subset \(C\) of \(X\) and any closed subset \(D\) disjoint from \(C\), then there exists an open set \(U\) with \(C \subset Cl U\) and \(Cl U \cap D = \emptyset\) [5].

Let \(C = \{ (1/(2n - 1), 0) \mid n \in Z \}, D = \{ (1/2n, 0) \mid n \in Z \} \cup \{\infty\}\), and \(U\) any open set with \(C \subset Cl U\). If \(U \not\subset \{ (1/2n, -1/m) \mid m, n \in Z \}\) then clearly \(Cl U \cap D = \emptyset\). If infinitely many elements of \(U\) are contained in \(\{ (1/2i, -1/m) \mid m \in Z \}\) for some \(i\), then \((1/2i, 0) \in Cl U\). On the other hand, if \(U \subset \{ (1/2n, -1/m) \mid m, n \in Z \}\) and \(U \cap \{ (1/2i, -1/m) \mid m \in Z \}\) is finite for each \(i \in Z\), then \(\infty \in Cl U\). Hence \(X\) is not C-compact.

2. Characterizations of C-compact and functionally compact spaces.

**Definitions.** (a) A Hausdorff space \(X\) is functionally compact if for every open filter \(\mathcal{U}\) in \(X\) such that the intersection \(A\) of the elements of \(\mathcal{U}\) equals the intersection of the closure of the elements of \(\mathcal{U}\), then \(\mathcal{U}\) is the neighborhood filter of \(A\).

(b) A closed subset \(C\) of a space \(X\) is regular closed if for any \(x \in C\) there exists an open neighborhood \(U_x\) with \(Cl U_x \cap C = \emptyset\).

(c) Let \(S^c\) be a subset of a space \(X\). An open cover \(\{U_a\}_{a \in A}\) of \(S^c\) will be said to be a regular cover if \(X \setminus \bigcup_{a \in A} U_a\) is a regular closed set.

(d) A space \(X\) is regular seminormal if given a regular closed set \(C\) of \(X\) and an open set \(V\) containing \(C\), then there exists a regular open set \(R\) with \(C \subset R \subset V\).

**Theorem 1.** The following properties are equivalent.

(i) \(X\) is functionally compact.

(ii) Every continuous function from \(X\) into any Hausdorff space is closed.
(iii) Given a regular closed subset C of X, an open cover $\mathcal{B}$ of $X \setminus C$, and an open neighborhood $U$ of $C$, then there exist $O_i \in \mathcal{B}$, $1 \leq i \leq n$, such that $X = U \cup \text{Cl}_X \bigcup_{i=1}^{n} O_i$.

(iv) Given an open regular cover $\mathcal{B}$ of any closed set $C$, then there exist $O_i \in \mathcal{B}$, $1 \leq i \leq n$ such that $C \subset \text{Cl}_X \bigcup_{i=1}^{n} O_i$.

(v) $X$ is absolutely closed and regular seminormal.

Proof. The equivalence of (i) and (ii) has been shown by Dickman and Zame [1]. The statement that (i) and (iii) are equivalent is in [3]. (iii) $\Rightarrow$ (iv). Let $\mathcal{B} = \{O_a\}_{a \in A}$ be a regular open cover for the closed set $C$. Then $D = X \setminus \bigcup_{a \in A} O_a$ is regular closed, $\mathcal{B}$ is a cover of $X \setminus D$, and $X \setminus C$ is an open neighborhood of $D$. Hence there exist $O_i \in \mathcal{B}$, $1 \leq i \leq n$, such that $X = (X \setminus C) \cup \text{Cl}_X \bigcup_{i=1}^{n} O_i$; that is, $C \subset \text{Cl}_X \bigcup_{i=1}^{n} O_i$. A similar argument shows (iv) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (v). That $X$ is absolutely closed may be seen by choosing in (iii) the empty set for both $C$ and $U$. We show $X$ is regular seminormal. Consider a regular closed set $C$ and open set $V$ containing $C$. For each $x \in X \setminus C$ choose a neighborhood $U_x$ with $\text{Cl} U_x \cap C = \emptyset$. By (iii) there exist $U_{x_i}$, $1 \leq i \leq n$, with $C \subset X \setminus \text{Cl}_X \bigcup_{i=1}^{n} O_i \subset V$, and clearly $X \setminus \text{Cl}_X \bigcup_{i=1}^{n} O_i$ is regular open.

(v) $\Rightarrow$ (iii). Let an open cover $\mathcal{B}$ of $X \setminus C$ be given where $C$ is regular closed. Let $U$ be a neighborhood of $C$. Choose a regular open set $R$ and $O_i \in \mathcal{B}$, $1 \leq i \leq n$, such that $C \subset R \subset U$ and $X = \text{Cl} R \setminus R \subset \text{Cl} \bigcup_{i=1}^{n} O_i$. Since $R$ is a regular open set we have $\text{Cl} R \setminus R \subset \text{Cl} \bigcup_{i=1}^{n} O_i$, so that $X = U \cup \text{Cl} \bigcup_{i=1}^{n} O_i$.

Definition (e). A Hausdorff space $(X, \tau)$ is rim C-compact (rim functionally compact) if there exists a neighborhood system for each point of $X$ consisting of open sets, $V$, with the property that given a closed set $Q$ of $\text{Cl} V \setminus V$ and a $\tau$-open cover (regular cover) $\mathcal{Y}$ of $Q$, then there exists $V_i \in \mathcal{Y}$; $1 \leq i \leq n$, with $Q \subset \text{Cl}_X \bigcup_{i=1}^{n} V_i$.

Theorem 2. A Hausdorff space $X$ is C-Compact (functionally compact) if and only if it is absolutely closed and rim C-compact (rim functionally compact).

Proof. A C-compact space is clearly absolutely closed and rim C-compact. To prove the converse we use the following obvious characterization of C-compactness: $X$ is C-compact if and only if given any open cover $\mathcal{Y}$ of $X$, and any $V \in \mathcal{Y}$ then there exist $V_i \in \mathcal{Y}$; $1 \leq i \leq n$, with $X \subset V \cup \text{Cl} \bigcup_{i=1}^{n} V_i$. Let then $V \in \mathcal{Y}$ where $\mathcal{Y}$ is an open cover of $X$. Choose for each $x \in V$ a rim C-compact neighborhood $V_x$ with $V_x \subset V$. Select from the cover $(\mathcal{Y} \setminus \{V\}) \cup \{V_x | x \in V\}$ elements.
Let $V_i \in \mathcal{V}\backslash\{V\}$, $1 \leq i \leq n$, and $V_{x_i} \in \{V_x \mid x \in V\}$, $1 \leq i \leq m$, with

$$X = Cl\left(\bigcup_{i=1}^{n} V_i \cup \bigcup_{i=1}^{m} V_{x_i}\right).$$

Let $V_{y_i}^{(i)} \in \mathcal{V}\backslash\{V\}$, $1 \leq J \leq n_i$, be such that $Cl\ V_{x_i} \backslash V \subset Cl\ \bigcup_{i=1}^{m} V_{y_i}^{(i)}$, $1 \leq i \leq m$. Then

$$X = V \cup Cl\left(\bigcup_{i=1}^{n} V_i \cup \bigcup_{1 \leq i \leq m} \bigcup_{1 \leq J \leq n_i} V_{y_i}^{(i)}\right).$$

A functionally compact space is clearly absolutely closed and rim functionally compact. The converse may be proved by using Theorem 1 (iii) and an argument similar to the one given above.

3. Embeddings. The absolute closed extensions constructed in the following theorem are of the type described by Fomin [2].

**Theorem 3.** Any rim C-compact (rim functionally compact) space can be embedded as a dense subspace of a C-compact (functionally compact) space.

**Proof.** Let $X$ be rim C-compact (rim functionally compact). Let $\mathcal{B}$ denote the base of all open sets whose boundaries are C-compact (functionally compact). Let $X = \{\xi_x \mid x \in X\}$ where $\xi_x = \{0 \in \mathcal{B} \mid x \in 0\}$ and let $Y$ denote the set of all maximal $\mathcal{B}$-filters with empty adherent set. Topologize $E = X \cup Y$ as follows. Let a neighborhood system for an element $\xi \in E$ be composed of all sets of the form $O_\xi = \{\xi' \in E \mid O \in \xi'\}$ for $O \in \xi$. Since $\mathcal{B}$ is a complemented base, that is $\bar{O} \in \mathcal{B}$ for $O \in \mathcal{B}$, we have that $E$ is an absolutely closed extension of $X$ [2]. To show that $E$ is C-compact (functionally compact) it is sufficient, by §2 Theorem 2, to show that $E$ is rim C-compact (rim functionally compact). To do this we note that for any open set $0$ of $X$, $Cl_{E}O_{E} \backslash O_{E} = O_{E} \cup \{\xi_x \mid x \in Cl_{x}O\}$, so that $\{O_{E} \mid O \in \mathcal{B}\}$ is an open base for $E$ with boundary of $O_{E}$ C-compact (functionally compact).

**References**


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