Pacific Journal of Mathematics

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Vol. 37, No. 3 March 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 3, 1971

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G. GOSS AND G. VIGLINO

In the first section of this note a question posed by G. Viglino is resolved by constructing a C-compact space which is not seminormal. In the second section some characterizations of C-compact and functionally compact spaces are introduced. In the final section, embedding theorems of spaces into C-compact and functionally compact spaces are noted.

1. A Counterexample.

DEFINITIONS. (a) A Hausdorff space X is *absolutely closed* if given an open cover $\mathscr{V}$ of X, then there exists a finite number of elements of $\mathscr{V}$, say V_i, $1 \leqq i \leqq n$, with $X \subset \mathrm{Cl} \bigcup_{i=1}^{n} V_i$.

(b) A Hausdorff space (X, τ) is *C-compact* if given a closed set Q of X and a τ-open cover $\mathscr{V}$ of Q, then there exists a finite number of elements of $\mathscr{V}$, say V_i, $1 \leqq i \leqq n$, with $Q \subset Cl_X \bigcup_{i=1}^{n} V_i$.

(c) An open set V is *regular* if $V = \bar{V}^\circ$.

(d) A space X is *seminormal* if given a closed subset C of X and an open set V containing G, then there exists a regular open set R with $C \subset R \subset V$.

G. Viglino has shown that a seminormal absolutely closed space is C-compact, and posed the question as to whether or not the converse holds [5]. The following is an example of a C-compact space which is not seminormal. An example has also been obtained by T. Lominac, Abstract #682-54-33.

EXAMPLE. Let Z represent the set of positive integers. Let

$$X = \left\{\left(\frac{1}{2n-1}, \frac{1}{m}\right) \middle| n, m \in Z\right\} \cup \left\{\left(\frac{1}{2n}, -\frac{1}{m}\right) \middle| n, m \in Z\right\} \cup \left\{\left(\frac{1}{n}, 0\right) \middle| n \in Z\right\} \cup \{\infty\} .$$

Topologize X as follows. Partition Z into infinitely many infinite equivalence classes, $\{Z_i\}_{i=1}^{\infty}$, and let $\{Z_1^J\}_{J=1}^{\infty}$ be a partition of Z_1 into infinitely many infinite equivalent classes. Let Φ denote a bijection from $\{(1/(2n-1), 1/m) \mid n, m \in Z\}$ to $Z\backslash\{1\}$. Let a neighborhood system for the points of the form $(1/(2i-1), 0)$ be composed of all sets of the form $U_{(2i-1,0;k)} = V \cup F$ where $V = \{(1/(2i-1), 0)\} \cup \{(1/(2i-1), 1/m) \mid m \geqq k\}$ and $F = \{(1/(2n-1), 1/m) \mid m \in Z_i \text{ and } n \geqq k\} \cup \{(1/2n, -1/s) \mid$

$s \in \bigcup_{m \in Z_i} Z_{\Phi(1/(2n-1),1/m)}$ and $h \geqq k\}$ for some $k \in Z$. Let a system for $(1/(2i-1), 1/J)$ be the sets

$$\bigcup_{(2i-1,J;k)} \left\{\left(\frac{1}{2i-1}, \frac{1}{J}\right)\right\} \cup \left\{\left(\frac{1}{2n}, -\frac{1}{s}\right) \middle| s \in Z_{\Phi(1/(2i-1),1/J)} \text{ and } n \geqq k\right\}$$

for some $k \in Z$. The points $(1/2n, -1/m)$ are to be open, and a neighborhood system for the points of the form $(1/2i, 0)$ consists of sets of the form $\bigcup_{(2i,0;k)} = V \cup F$ where $V = \{(1/2i, 0)\} \cup \{(1/2i, -1/m) \mid m \geqq k\}$ and $F = \{(1/2n, -1/m) \mid m \in Z_1^i$ and $n \geqq k\}$ for some $k \in Z$. Finally, let a system for the point ∞ be composed of those sets of the form $X \backslash T$, where $T = \{(1/n, 0) \mid n \in Z\} \cup \bigcup_{i=1}^{k} (\bigcup_{(2i-1,0,k)} \cup \bigcup_{(2i,0;k)})$ for $k \in Z$.

An argument similar to that given in Example 1 of [4] may be used to show that X is a C-compact topological space. To show X is not seminormal we use the following characterization: X is seminormal if and only if given any closed subset C of X and any closed subset D disjoint from C, then there exists an open set U with $C \subset Cl\, U$ and $Cl\, U \cap D = \varnothing$ [5].

Let $C = \{(1/(2n-1), 0) \mid n \in Z\}$, $D = \{(1/2n, 0) \mid n \in Z\} \cup \{\infty\}$, and U any open set with $C \subset Cl\, U$. If $U \not\subset \{(1/2n, -1/m) \mid m, n \in Z\}$ then clearly $Cl\, U \cap D \neq \varnothing$. If infinitely many elements of U are contained in $\{(1/2i, -1/m) \mid m \in Z\}$ for some i, then $(1/2i, 0) \in Cl\, U$. On the other hand, if $U \subset \{(1/2n, -1/m) \mid m, n \in Z\}$ and $U \cap \{(1/2i, -1/m) \mid m \in Z\}$ is finite for each $i \in Z$, then $\infty \in Cl\, U$. Hence X is not C-compact.

2. Characterizations of C-compact and functionally compact spaces.

DEFINITIONS. (a) A Hausdorff space X is *functionally compact* if for every open filter $\mathscr{U}$ in X such that the intersection A of the elements of $\mathscr{U}$ equals the intersection of the closure of the elements of $\mathscr{U}$, then $\mathscr{U}$ is the neighborhood filter of A.

(b) A closed subset C of a space X is *regular closed* if for any $x \notin C$ there exists an open neighborhood U_x with $Cl\, U_x \cap C = \varnothing$.

(c) Let S be a subset of a space X. An open cover $\{U_\alpha\}_{\alpha \in A}$ of S will be said to be a *regular cover* if $X \backslash \bigcup_{\alpha \in A} U_\alpha$ is a regular closed set.

(d) A space X is *regular seminormal* if given a regular closed set C of X and an open set V containing C, then there exists a regular open set R with $C \subset R \subset V$.

THEOREM 1. *The following properties are equivalent.*

(i) *X is functionally compact.*

(ii) *Every continuous function from X into any Hausdorff space is closed.*

(iii) *Given a regular closed subset C of X, an open cover $\mathscr{B}$ of $X \backslash C$, and an open neighborhood U of C, then there exist $O_i \in \mathscr{B}$, $1 \leqq i \leqq n$, such that $X = U \cup Cl_X \bigcup_{i=1}^n O_i$.*

(iv) *Given an open regular cover $\mathscr{B}$ of any closed set C, then* there exist *$O_i \in \mathscr{B}$, $1 \leqq i \leqq n$ such that $C \subset Cl_X \bigcup_{i=1}^n O_i$.*

(v) *X is absolutely closed and regular seminormal.*

Proof. The equivalence of (i) and (ii) has been shown by Dickman and Zame [1]. The statement that (i) and (iii) are equivalent is in [3]. (iii) $\Rightarrow$ (iv). Let $\mathscr{B} = \{O_\alpha\}_{\alpha \in A}$ be a regular open cover for the closed set C. Then $D = X \backslash \bigcup_{\alpha \in A} O_\alpha$ is regular closed, $\mathscr{B}$ is a cover of $X \backslash D$, and $X \backslash C$ is an open neighborhood of D. Hence there exist $O_i \in \mathscr{B}$, $1 \leqq i \leqq n$, such that $X = (X \backslash C) \cup Cl_X \bigcup_{i=1}^n O_i$; that is, $C \subset Cl_X \bigcup_{i=1}^n O_i$. A similar argument shows (iv) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (v). That X is absolutely closed may be seen by choosing in (iii) the empty set for both C and U. We show X is regular seminormal. Consider a regular closed set C and open set V containing C. For each $x \in X \backslash C$ choose a neighborhood U_x with $Cl\, U_x \cap C = \varnothing$. By (iii) there exist U_{x_i}, $1 \leqq i \leqq n$, with $C \subset X \backslash Cl_X \bigcup_{i=1}^n O_i \subset V$, and clearly $X \backslash Cl_X \bigcup_{i=1}^n O_i$ is regular open.

(v) $\Rightarrow$ (iii). Let an open cover $\mathscr{B}$ of $X \backslash C$ be given where C is regular closed. Let U be a neighborhood of C. Choose a regular open set R and $O_i \in \mathscr{B}$, $1 \leqq i \leqq n$, such that $C \subset R \subset U$ and $X = Cl \bigcup_{i=1}^n O_i$. Since R is a regular open set we have $Cl\, R \backslash R \subset Cl \bigcup_{i=1}^n O_i$ so that $X = U \cup Cl \bigcup_{i=1}^n O_i$.

DEEINITION (e). A Hausdorff space (X, τ) is *rim C-compact* (*rim functionally compact*) if there exists a neighborhood system for each point of X consisting of open sets, V, with the property that given a closed set Q of $Cl\, V \backslash V$ and a τ-open cover (regular cover) $\mathscr{V}$ of Q, then there exists $V_i \in \mathscr{V}$, $1 \leqq i \leqq n$, with $Q \subset Cl_X \bigcup_{i=1}^n V_i$.

THEOREM 2. *A Hausdorff space X is C-Compact* (functionally compact) *if and only if it is absolutely closed and rim C-compact* (rim functionally compact).

Proof. A C-compact space is clearly absolutely closed and rim C-compact. To prove the converse we use the following obvious characterization of C-compactness: X is C-compact if and only if given any open cover $\mathscr{V}$ of X, and any $V \in \mathscr{V}$ then there exist $V_i \in \mathscr{V}$, $1 \leqq i \leqq n$, with $X \subset V \cup Cl \bigcup_{i=1}^n V_i$. Let then $V \in \mathscr{V}$ where $\mathscr{V}$ is an open cover of X. Choose for each $x \in V$ a rim C-compact neighborhood V_x with $V_x \subset V$. Select from the cover $(\mathscr{V} \backslash \{V\}) \cup \{V_x \mid x \in V\}$ elements

$V_i \in \mathscr{V}\backslash\{V\}$, $1 \leqq i \leqq n$, and $V_{x_i} \in \{V_x | x \in V\}$, $1 \leqq i \leqq m$, with

$$X = Cl\left(\bigcup_{i=1}^{n} V_i \cup \bigcup_{i=1}^{m} V_{x_i}\right).$$

Let $V_J^{(i)} \in \mathscr{V}\backslash\{V\}$, $1 \leqq J \leqq n_i$, be such that $Cl\, V_{x_i}\backslash V \subset Cl \bigcup_{J=1}^{n_i} V_J^{(i)}$, $1 \leqq i \leqq m$. Then

$$X = V \cup Cl\left(\bigcup_{i=1}^{n} V_i \cup \bigcup_{\substack{1 \leqq i \leqq m \\ 1 \leqq J \leqq n_i}} V_J^{(i)}\right).$$

A functionally compact space is clearly absolutely closed and rim functionally compact. The converse may be proved by using Theorem 1 (iii) and an argument similar to the one given above.

3. Embeddings. The absolute closed extensions constructed in the following theorem are of the type described by Fomin [2].

THEOREM 3. *Any rim C-compact* (rim functionally compact) *space can be embedded as a dense subspace of a C-compact* (functionally compact) *space.*

Proof. Let X be rim C-compact (rim functionally compact). Let $\mathscr{B}$ denote the base of all open sets whose boundaries are C-compact (functionally compact). Let $X = \{\xi_x | x \in X\}$ where $\xi_x = \{0 \in \mathscr{B} | x \in 0\}$ and let Y denote the set of all maximal $\mathscr{B}$-filters with empty adherent set. Topologize $E = X \cup Y$ as follows. Let a neighborhood system for an element $\xi \in E$ be composed of all sets of the form $O_E = \{\xi' \in E | O \in \xi'\}$ for $O \in \xi$. Since $\mathscr{B}$ is a complemented base, that is $\bar{O}^c \in \mathscr{B}$ for $O \in \mathscr{B}$, we have that E is an absolutely closed extension of X [2]. To show that E is C-compact (functionally compact) it is sufficient, by § 2 Theorem 2, to show that E is rim C-compact (rim functionally compact). To do this we note that for any open set 0 of X, $Cl_E\, O_E\backslash O_E = O_E \cup \{\xi_x | x \in Cl_X O\}$, so that $\{O_E | O \in \mathscr{B}\}$ is an open base for E with boundary of O_E C-compact (functionally compact).

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Received February 18, 1971.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 37, No. 3 March, 1971