ARCWISE CONNECTIVITY OF SEMI-APOSYNDETC PLANE CONTINUA

CHARLES LEMUEL HAGOPIAN
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CHARLES L. HAGOPIAN

Suppose $M$ is a bounded semi-aposyndetic plane continuum and for any positive real number $\varepsilon$ there are at most a finite number of complementary domains of $M$ of diameter greater than $\varepsilon$. In this paper it is proved that $M$ is arcwise connected.

Let $M$ be a continuum (a closed connected point set) and let $x$ and $y$ be distinct points of $M$. If $M$ contains a continuum $H$ and an open set $G$ such that $x \in G \subset H \subset M - \{y\}$, then $M$ is said to be aposyndetic at $x$ with respect to $y$ [4]. $M$ is said to be semi-aposyndetic if for each pair of distinct points $x$ and $y$ of $M$, $M$ is aposyndetic either at $x$ with respect to $y$ or at $y$ with respect to $x$. In [3] it is proved that every bounded semi-aposyndetic plane continuum which does not have infinitely many complementary domains is arcwise connected. For other results concerning semi-aposyndetic plane continua see [1] and [2].

Let $x$ and $y$ be distinct points of a metric space $S$. A finite collection $\{A_1, A_2, \ldots, A_m\}$ of sets in $S$ is a chain in $S$ from $x$ to $y$ provided $A_1$ contains $x$, $A_m$ contains $y$, and for $i$ and $j$ belonging to $\{1, 2, \ldots, m\}$, $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. If each element of a chain $\mathcal{A}$ has diameter less than $r$ (a positive real number) then $\mathcal{A}$ is said to be an $r$-chain. Suppose $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ are chains in $S$ from $x$ to $y$. The chain $\mathcal{B}$ is said to run straight through $\mathcal{A}$ provided the closure of each element of $\mathcal{B}$ is contained in an element of $\mathcal{A}$ and if $B_i$ and $B_k$ ($1 \leq i \leq k \leq n$) both lie in an element $A_s$ of $\mathcal{A}$, then for each integer $j$ ($i < j < k$), $B_j$ is contained in an element of $\mathcal{A}$ whose intersection with $A_s$ is nonvoid.

If $M$ is a bounded plane continuum and for any positive real number $\varepsilon$ there are at most a finite number of complementary domains of $M$ of diameter greater than $\varepsilon$, then $M$ is said to be an $E$-continuum [6, p. 112].

The boundary of a set $A$ is denoted by $\text{Bd } A$.

**THEOREM 1.** Suppose $M$ is a semi-aposyndetic $E$-continuum is $S$ (a 2-sphere with metric $\varphi$), $U$ is a disk in $S$, $x$ and $y$ are distinct points which belong to the same component of $M \cap U$, and $V$ is an open disk in $S$ containing $U$. Then for any positive real number $r$ less than both $\varphi (x, y)/5$ and $\varphi (\text{Bd } U, \text{Bd } V)/5$ there exists an $r$-chain $\{H_1, H_2, \ldots, H_n\}$ ($n > 3$) in $S$ from $x$ to $y$ such that for each positive
integer i less than or equal n, \( H_i \) is a continuum in \( M \cap V \) and \( \varphi(H_i, \text{Bd } V) > 4r \).

**Proof.** Let \( G \) be the union of all components of \( S - M \) which have diameter less than \( r/3 \). Since \( M \) is a semi-aposyndetic \( E \)-continuum, \( M \cup G \) is a semi-aposyndetic continuum which does not have infinitely many complementary domains \([5, \text{ Th. 2 (proof)}]\). Let \( F \) be the \( x \)-component of \( U \cap (M \cup G) \). \( F \) is a semi-aposyndetic continuum in \( S \) which does not have infinitely many complementary domains \([3, \text{ Th. 1}]\) \((D \text{ and } M \text{ in } [3] \text{ are } S - U \text{ and } M \cup G \text{ respectively})\). Hence \( F \) is arcwise connected \([3, \text{ Th. 2}]\). Let \( A \) be an arc in \( F \) from \( x \) to \( y \). There exists a finite point set \( B \) in \( A - \{x, y\} \) such that each component of \( A - B \) has diameter less than \( r/3 \). For each component \( C \) of \( A - B \), let \( G(C) \) be \( C \) union all components of \( G \) which intersect \( C \) and let \( Z(C) \) be the boundary (relative to \( S \)) of \( G(C) \). For each component \( C \) of \( A - B \), since the boundary of each component of \( G \) is a continuum \([6, \text{ Th. 2.1, p. 105}]\) and each point of \( C \) that is not in \( G \) belongs to \( Z(C) \), \( Z(C) \) is a continuum of diameter less than \( r \) in \( M \).

Let \( \mathcal{K} \) be the finite coherent collection of continua \( \{Z(C) | C \text{ is a component of } A - B \} \). The points \( x \) and \( y \) each belong to an element of \( \mathcal{K} \) and each element of \( \mathcal{K} \) intersects \( U \). It follows that any chain from \( x \) to \( y \) whose elements are members of \( \mathcal{K} \) has the specified conditions.

**Theorem 2.** If \( M \) is a semi-aposyndetic \( E \)-continuum, then \( M \) is arcwise connected.

**Proof.** Let \( S \) be a 2-sphere which contains \( M \) and let \( \varphi \) be a distance function on \( S \). Let \( p \) and \( q \) be distinct points of \( M \). Define \( r_i \) to be a positive real number less than both 1/8 and \( \varphi(p, q)/5 \) and let \( s_i = 4r_i \). According to Theorem 1, there exists an \( r_i \)-chain \( \{H_i, H_2^i, \ldots, H_n^i\} \) in \( S \) from \( p \) to \( q \) such that for each positive integer \( i \) less than or equal \( n \), \( H_i \) is a continuum in \( M \). Let \( m_i \) be the smallest integer greater than or equal to \( (n_i - 1)/2 \). There exist a set of disks \( \{U_i^1, U_i^2, \ldots, U_i^{m_i}\} \) and a set of open disks \( \{V_i^1, V_i^2, \ldots, V_i^{m_i}\} \) such that \( \{V_i^1, V_i^2, \ldots, V_i^{m_i}\} \) is an \( s_i \)-chain in \( S \) from \( p \) to \( q \) and for each positive \( i \) less than or equal \( m_i \), \( H_i^{1/2} \cup H_i^{1/2} \cup H_i^{1/2+1} \subset U_i \subset V_i^i \) (if \( n_i \) is even, let \( H_i^{1/2+1} = \emptyset \)).

Let \( \{p_1^1, p_1^2, \ldots, p_1^{m_1+1}\} \) be a point set such that \( p_1^1 = p, p_1^{m_1+1} = q \), and for each positive integer \( i \) less than or equal \( m_i \), \( p_i^i \) belongs to \( H_i^{1/2-1} \). Let \( t_i \) be the smallest number in the set \( \{\varphi(\text{Bd } U_i^1, \text{Bd } V_i^i | i \leq m_i \} \cup \{\varphi(p_i^i, p_i^{i+1}) | i \leq m_i \} \). Let \( r_2 \) be a positive real number less than both \( t_i/5 \) and 1/16. Define \( s^2 = 4r_2 \). For each positive in-
integer $i$ less than or equal $m$, there exists an $r_2$-chain $C_i$ in $S$ from $p_i$ to $p_{i+1}$ such that each element of $C_i$ is a continuum in $M \cap V_i$ and at a distance greater than $4r_2$ from $\text{Bd} \ V_i$ (Theorem 1). There exists an $r_2$-chain $\{H_1^i, H_2^i, \cdots, H_{m^2_2}^i\}$ in $S$ from $p$ to $q$ whose elements belong to $\bigcup_{i=1}^{m_2} C_i$ such that for each positive integer $i$ less than or equal $m$, $C_i \cap \{H_1^i, H_2^i, \cdots, H_{m^2_2}^i\}$ is a coherent collection. Let $m_2$ be the smallest integer greater than or equal to $(n_2 - 1)/2$. There exist a set of disks $\{U_1^i, U_2^i, \cdots, U_{m_2}^i\}$ and a set of open disks $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$ such that $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$ is an $s_2$-chain in $S$ from $p$ to $q$ and for each positive integer $i$ less than or equal $m_2$, $H_{j_2}^i \cup H_{j_2}^i \cup H_{j_2}^i \subset U_{j_2}^i \subset V_{j_2}^i$ (if $n_2$ is even, let $H_{j_2}^i = \emptyset$). Note that $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$ runs straight through $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$.

Continue this process. For $i = 3, 4, 5, \cdots$, there exists a chain $\{H_1^i, H_2^i, \cdots, H_{m^2_2}^i\}$ in $S$ from $p$ to $q$ whose elements are continua in $M$, and there exists an $s_3$-chain $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$ such that $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$ runs straight through $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$. The limiting set $L$ of the sequence $L_1, L_2, L_3, \cdots$ is a continuum in $M$ containing $p$ and $q$. Note that for each positive integer $i$, $L$ is contained in $\bigcup_{j=1}^{m_2} V_j^i$.

Let $x$ be a point of $L - \{p, q\}$. For each positive integer $i$, let $V_{j_2}^i$ be an element of $\{V_1^i, V_2^i, \cdots, V_{m_2}^i\}$ which contains $x$. Assume without loss of generality that $4 < j_1 < m_1 - 4$. For each positive integer $i$, let $P_i$ be $\{V_1^i, V_2^i, \cdots, V_{j_2}^i\}$ and let $F_i$ be $\{V_{j_2}^i, V_{j_2+1}^i, \cdots, V_{m_2}^i\}$. Let $P = \bigcup_{i=1}^{m_1} (P_i \cap L)$ and $F = \bigcup_{i=1}^{m_1} (F_i \cap L)$. $P$ and $F$ are nonempty disjoint relatively open subsets of $L$ and $P \cup F = L - \{x\}$. Hence $x$ is a separating point of $L$. It follows that $L$ has only two nonseparating points. Therefore $L$ is an arc [6, Th. 6.2, p. 54]. Hence $M$ is arcwise connected.

REMARK. Using [3, Th. 1] and Theorem 2 one can easily prove that if $M$ is a semi-aposyndetic $E$-continuum, then $M$ has Jones's cyclic property (that is, if $p$ and $q$ are distinct points of $M$ and no point cuts $p$ from $q$ in $M$, then there exists a simple closed curve lying in $M$ which contains $p$ and $q$).

REFERENCES


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