

Pacific Journal of Mathematics

**STRATIFIABLE SPACES, SEMI-STRATIFIABLE SPACES, AND
THEIR RELATION THROUGH MAPPINGS**

DAVID MICHAEL HENRY

STRATIFIABLE SPACES, SEMI-STRATIFIABLE SPACES, AND THEIR RELATION THROUGH MAPPINGS

MICHAEL HENRY

It is shown that the image of a stratifiable space under a pseudo-open compact mapping is semi-stratifiable. By strengthening the mapping from compact to finite-to-one the following results are also obtained. The image of a semi-stratifiable (semi-metric) space under an open finite-to-one mapping is semi-stratifiable (semi-metric).

Notation and terminology will follow that of Dugundji [6]. By a neighborhood of a set A , we will mean an open set containing A , and all mappings will be continuous and surjective.

DEFINITION 1.1. A topological space X is a *stratifiable space* if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that

- (a) $\bar{U}_n \subset U$,
- (b) $U_{n=1}^{\infty} U_n = U$,
- (c) $U_n \subset V_n$ whenever $U \subset V$.

DEFINITION 1.2. A topological space X is a *semi-stratifiable space* if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that

- (a) $U_{n=1}^{\infty} U_n = U$,
- (b) $U_n \subset V_n$ whenever $U \subset V$.

Ceder [3] introduced M_3 -spaces and Borges [2] renamed them "stratifiable", while Creede [4] studied semi-stratifiable spaces. A correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a *stratification* (semi-stratification) for the space X whenever it satisfies the conditions of Definition 1.1 (1.2).

LEMMA 1.3. *A space X is stratifiable if and only if to each closed subset $F \subset X$ one can assign a sequence $\{U_n\}$ of open subsets of X such that*

- (a) $F \subset U_n$ for each n ,
- (b) $\bigcap_{n=1}^{\infty} \bar{U}_n = F$,
- (c) $U_n \subset V_n$ whenever $U \subset V$.

LEMMA 1.4. *A space X is semi-stratifiable if and only if to each closed set $F \subset X$ one can assign a sequence $\{U_n\}$ of open subsets*

of X such that

- (a) $F \subset U_n$ for each n ,
- (b) $\bigcap_{n=1}^{\infty} U_n = F$
- (c) $U_n \subset V_n$ whenever $U \subset V$.

A correspondence $F \rightarrow \{U_n\}_{n=1}^{\infty}$ is a *dual stratification* (semi-stratification) for the space X whenever it satisfies the three conditions of Lemma 1.3 (1.4). For convenience in the proofs which will be encountered, each member in the range of a correspondence will also be called a dual stratification (semi-stratification) of the closed set to which it is associated.

2. Mappings from stratifiable spaces. We now exhibit a natural way in which semi-stratifiable spaces may arise.

DEFINITION 2.1. A mapping $f: X \rightarrow Y$ is *pseudo-open* if for each $y \in Y$ and any neighborhood U of $f^{-1}(y)$, it follows that $y \in \text{int}[f(U)]$.

DEFINITION 2.2. A mapping $f: X \rightarrow Y$ is *compact* if $f^{-1}(y)$ is compact for each $y \in Y$.

THEOREM 2.3. *If X is stratifiable and $f: X \rightarrow Y$ is a pseudo-open compact mapping, then Y is semi-stratifiable.*

Proof. Let $F \subset Y$ be a closed set. Then $f^{-1}(F)$ is closed in X and, hence, by Lemma 1.3, has a dual stratification $\{U_n\}$. We will show that the correspondence $F \rightarrow \{\text{int}[f(U_n)]\}$ is a dual semi-stratification for Y by proving that the collections $\{\text{int}[f(U_n)]\}$ satisfy the requirements of Lemma 1.4.

Part (c) of Lemma 1.4 is easily shown to be satisfied. For if F and G are closed subsets of Y such that $F \subset G$, then $f^{-1}(F) \subset f^{-1}(G)$, and denoting the dual stratifications of $f^{-1}(F)$ and $f^{-1}(G)$ by $\{U_n\}$ and $\{V_n\}$, respectively, we must have by Lemma 1.3(c) that $U_n \subset V_n$ for each n . Therefore, $\text{int}[f(U_n)] \subset \text{int}[f(V_n)]$.

With regard to part (a), it follows that $F \subset \text{int}[f(U_n)]$ for each n . This is because each U_n is a neighborhood of $f^{-1}(y)$ for every $y \in F$, and therefore $y \in \text{int}[f(U_n)]$ for every $y \in F$ by hypothesis of f being a pseudo-open mapping.

All that remains to be shown is that $\bigcap_{n=1}^{\infty} \text{int}[f(U_n)] = F$, and this will verify (b). From the preceding paragraph we know that $F \subset \bigcap_{n=1}^{\infty} \text{int}[f(U_n)]$. To get inclusion in the reverse direction, assume $z \in \bigcap_{n=1}^{\infty} \text{int}[f(U_n)]$. Then $z \in \text{int}[f(U_n)]$ for every n ; hence, there exist points $x_n \in U_n$ such that $f(x_n) = z$. Since f is a compact mapping, the sequence $\{x_n\}$ has an accumulation point x . Therefore, given any

neighborhood V of x , there exist infinitely many integers n_i such that $x_{n_i} \in V$. Thus, V has a nonempty intersection with infinitely many U_n , and since we may assume that the collection $\{U_n\}$ is descending, this implies that $V \cap U_n \neq \emptyset$ for every n . That is, $x \in \bigcap_{n=1}^{\infty} \bar{U}_n$. But $\{U_n\}$ was a dual stratification for $f^{-1}(F)$ which implies that $\bigcap_{n=1}^{\infty} \bar{U}_n = f^{-1}(F)$. Thus, $x \in f^{-1}(F)$ and $f(x) \in F$. Furthermore, $f(x) = z$ because $x \in \{\bar{x}_n\}$ and $\{\bar{x}_n\} \subset \overline{f^{-1}(z)} = f^{-1}(z)$. Hence, $z \in F$ and the proof is complete.

COROLLARY 2.4. *If X is a stratifiable space and $f: X \rightarrow Y$ is an open compact mapping, then Y is a metacompact semi-stratifiable space.*

Proof. The image of a paracompact space under an open compact mapping is metacompact by Theorem 4 of [1]. Since open mappings are pseudo-open, Y is also semi-stratifiable.

If the converse of Theorem 2.3 is true, then another characterization of semi-stratifiable spaces is available. Also, Corollary 2.4 is an analogue of the well-known result that an open compact image of a metric space is a space having a uniform base (metacompact and developable).

3. Mappings from semi-stratifiable and semi-metrizable spaces. Semi-stratifiable and semi-metrizable spaces are closely related in the sense that a first countable semi-stratifiable space is semi-metrizable, and conversely [4, Corollary 1.4]. Creede showed that semi-stratifiable spaces are preserved under closed mappings, but a similar result is not true for semi-metric spaces since there is no guarantee that the image will be first countable, even if the domain is a separable metric. Nor is the property of being semi-metrizable transmitted under an open mapping, for in this case, Creede [5, Theorem 3.4] has exhibited a non-semistratifiable Hausdorff space which is the open image of a separable metric space. However, by placing a suitable restriction on an open mapping, a class of open mappings can be found in which members preserve both semi-stratifiable and semi-metric spaces.

THEOREM 3.1. *If X is semi-stratifiable and $f: X \rightarrow Y$ is a pseudo-open finite-to-one mapping, then Y is semi-stratifiable.*

Proof. Let $F \subset Y$ be an arbitrary closed set. Then $f^{-1}(F)$ is closed in X and has a dual semi-stratification $\{U_n\}$. We will use

Lemma 1.4 to show that the correspondence $F \rightarrow \{\text{int}[f(U_n)]\}$ is a dual semi-stratification for Y .

Parts (a) and (c) are verified in the same manner as in the proof of Theorem 2.3. To verify (b), assume $z \in \bigcap_{n=1}^{\infty} \text{int}[f(U_n)]$. Then there exist points $x_n \in U_n$ such that $f(x_n) = z$ for every n . Since f is a finite-to-one mapping, there exists an integer m such that $x_m \in \bigcap_{n=1}^{\infty} U_n$. But $\bigcap_{n=1}^{\infty} U_n = f^{-1}(F)$ which implies that $x_m \in f^{-1}(F)$. Hence, $z \in F$ and the proof is complete.

COROLLARY 3.2. *The image of a semi-stratifiable space under an open finite-to-one mapping is semi-stratifiable.*

COROLLARY 3.3. *The image of a semi-metric space under an open finite-to-one mapping is semi-metrizable.*

REFERENCES

1. A. V. Arhangel'skii, *On mappings of metric spaces*, Soviet Math. Dokl., **3** (1962), 953-956.
2. C. Borges, *On stratifiable spaces*, Pacific J. Math., **17** (1966), 1-16.
3. J. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **11** (1961), 105-125.
4. G. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math., **32** (1970), 47-54.
5. ———, *Semi-stratifiable spaces and a factorization of a metrization theorem due to Bing*, Ph. D. thesis in Mathematics, Arizona State University, 1968.
6. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

Received August 26, 1970 and in revised form November 2, 1970. This research was supported by a TCU Research Fellowship and represents a portion of the author's doctoral dissertation, which was begun the direction of the late Professor H. Tamano and completed under D. R. Traylor and Howard Cook.

TEXAS CHRISTIAN UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 37, No. 3

March, 1971

Mohammad Shafqat Ali and Marvin David Marcus, <i>On the degree of the minimal polynomial of a commutator operator</i>	561
Howard Anton and William J. Pervin, <i>Integration on topological semifields</i>	567
Martin Bartelt, <i>Multipliers and operator algebras on bounded analytic functions</i>	575
Donald Earl Bennett, <i>Aposyndetic properties of unicoherent continua</i>	585
James W. Bond, <i>Lie algebras of genus one and genus two</i>	591
Mario Borelli, <i>The cohomology of divisorial varieties</i>	617
Carlos R. Borges, <i>How to recognize homeomorphisms and isometries</i>	625
J. C. Breckenridge, <i>Burkill-Cesari integrals of quasi additive interval functions</i>	635
J. Csima, <i>A class of counterexamples on permanents</i>	655
Carl Hanson Fitzgerald, <i>Conformal mappings onto ω-swirly domains</i>	657
Newcomb Greenleaf, <i>Analytic sheaves on Klein surfaces</i>	671
G. Goss and Giovanni Viglino, <i>C-compact and functionally compact spaces</i>	677
Charles Lemuel Hagopian, <i>Arcwise connectivity of semi-aposyndetic plane continua</i>	683
John Harris and Olga Higgins, <i>Prime generators with parabolic limits</i>	687
David Michael Henry, <i>Stratifiable spaces, semi-stratifiable spaces, and their relation through mappings</i>	697
Raymond D. Holmes, <i>On contractive semigroups of mappings</i>	701
Joseph Edmund Kist and P. H. Maserick, <i>BV-functions on semilattices</i>	711
Shūichirō Maeda, <i>On point-free parallelism and Wilcox lattices</i>	725
Gary L. Musser, <i>Linear semiprime $(p; q)$ radicals</i>	749
William Charles Nemitz and Thomas Paul Whaley, <i>Varieties of implicative semilattices</i>	759
Jaroslav Nešetřil, <i>A congruence theorem for asymmetric trees</i>	771
Robert Anthony Nowlan, <i>A study of H-spaces via left translations</i>	779
Gert Kjærgaard Pedersen, <i>Atomic and diffuse functionals on a C*-algebra</i>	795
Tilak Raj Prabhakar, <i>On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials</i>	801
Leland Edward Rogers, <i>Mutually aposyndetic products of chainable continua</i>	805
Frederick Stern, <i>An estimate for Wiener integrals connected with squared error in a Fourier series approximation</i>	813
Leonard Paul Sternbach, <i>On k-shrinking and k-boundedly complete basic sequences and quasi-reflexive spaces</i>	817
Pak-Ken Wong, <i>Modular annihilator A*-algebras</i>	825