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It is shown that the image of a stratifiable space under a pseudo-open compact mapping is semi-stratifiable. By strengthening the mapping from compact to finite-to-one the following results are also obtained. The image of a semi-stratifiable (semi-metric) space under an open finite-to-one mapping is semi-stratifiable (semi-metric).

Notation and terminology will follow that of Dugundji [6]. By a neighborhood of a set A, we will mean an open set containing A, and all mappings will be continuous and surjective.

DEFINITION 1.1. A topological space X is a *stratifiable space* if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that

(a) $\bar{U}_n \subset U$,
(b) $\bigcup_{n=1}^{\infty} U_n = U$,
(c) $U_n \subset V_n$ whenever $U \subset V$.

DEFINITION 1.2. A topological space X is a *semi-stratifiable space* if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that

(a) $\bigcup_{n=1}^{\infty} U_n = U$,
(b) $U_n \subset V_n$ whenever $U \subset V$.

Ceder [3] introduced M_3-spaces and Borges [2] renamed them "stratifiable", while Creede [4] studied semi-stratifiable spaces. A correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is a *stratification* (semi-stratification) for the space X whenever it satisfies the conditions of Definition 1.1 (1.2).

LEMMA 1.3. *A space X is stratifiable if and only if to each closed subset $F \subset X$ one can assign a sequence $\{U_n\}$ of open subsets of X such that*

(a) $F \subset U_n$ *for each* n,
(b) $\bigcap_{n=1}^{\infty} \bar{U}_n = F$,
(c) $U_n \subset V_n$ *whenever* $U \subset V$.

LEMMA 1.4. *A space X is semi-stratifiable if and only if to each closed set $F \subset X$ one can assign a sequence $\{U_n\}$ of open subsets* *of X such that*

(a) $F \subset U_n$ *for each* n,

(b) $\bigcap_{n=1}^{\infty} U_n = F$

(c) $U_n \subset V_n$ *whenever* $U \subset V$.

A correspondence $F \rightarrow \{U_n\}_{n=1}^{\infty}$ is a *dual stratification* (semi-stratification) for the space X whenever it satisfies the three conditions of Lemma 1.3 (1.4). For convenience in the proofs which will be encountered, each member in the range of a correspondence will also be called a dual stratification (semi-stratification) of the closed set to which it is associated.

2. **Mappings from stratifiable spaces.** We now exhibit a natural way in which semi-stratifiable spaces may arise.

DEFINITION 2.1. A mapping $f: X \rightarrow Y$ is *pseudo-open* if for each $y \in Y$ and any neighborhood U of $f^{-1}(y)$, it follows that $y \in \operatorname{int}[f(U)]$.

DEFINITION 2.2. A mapping $f: X \rightarrow Y$ is *compact if* $f^{-1}(y)$ is compact for each $y \in Y$.

THEOREM 2.3. *If X is stratifiable and $f: X \rightarrow Y$ is a pseudo-open compact mapping, then Y is semi-stratifiable.*

Proof. Let $F \subset Y$ be a closed set. Then $f^{-1}(F)$ is closed in X and, hence, by Lemma 1.3, has a dual stratification $\{U_n\}$. We will show that the correspondence $F \rightarrow \{\operatorname{int}[f(U_n)]\}$ is a dual semi-stratification for Y by proving that the collections $\{\operatorname{int}[f(U_n)]\}$ satisfy the requirements of Lemma 1.4.

Part (c) of Lemma 1.4 is easily shown to be satisfied. For if F and G are closed subsets of Y such that $F \subset G$, then $f^{-1}(F) \subset f^{-1}(G)$, and denoting the dual stratifications of $f^{-1}(F)$ and $f^{-1}(G)$ by $\{U_n\}$ and $\{V_n\}$, respectively, we must have by Lemma 1.3(c) that $U_n \subset V_n$ for each n. Therefore, $\operatorname{int}[f(U_n)] \subset \operatorname{int}[f(V_n)]$.

With regard to part (a), it follows that $F \subset \operatorname{int}[f(U_n)]$ for each n. This is because each U_n is a neighborhood of $f^{-1}(y)$ for every $y \in F$, and therefore $y \in \operatorname{int}[f(U_n)]$ for every $y \in F$ by hypothesis of f being a pseudo-open mapping.

All that remains to be shown is that $\bigcap_{n=1}^{\infty} \operatorname{int}[f(U_n)] = F$, and this will verify (b). From the preceding paragraph we know that $F \subset \bigcap_{n=1}^{\infty} \operatorname{int}[f(U_n)]$. To get inclusion in the reverse direction, assume $z \in \bigcap_{n=1}^{\infty} \operatorname{int}[f(U_n)]$. Then $z \in \operatorname{int}[f(U_n)]$ for every n; hence, there exist points $x_n \in U_n$ such that $f(x_n) = z$. Since f is a compact mapping, the sequence $\{x_n\}$ has an accumulation point x. Therefore, given any neighborhood V of x, there exist infinitely many integers n_i such that $x_{n_i} \in V$. Thus, V has a nonempty intersection with infinitely many U_n, and since we may assume that the collection $\{U_n\}$ is descending, this implies that $V \cap U_n \neq \phi$ for every n. That is, $x \in \bigcap_{n=1}^{\infty} \bar{U}_n$. But $\{U_n\}$ was a dual stratification for $f^{-1}(F)$ which implies that $\bigcap_{n=1}^{\infty} \bar{U}_n = f^{-1}(F)$. Thus, $x \in f^{-1}(F)$ and $f(x) \in F$. Furthermore, $f(x) = z$ because $x \in \overline{\{x_n\}}$ and $\overline{\{x_n\}} \subset \overline{f^{-1}(z)} = f^{-1}(z)$. Hence, $z \in F$ and the proof is complete.

COROLLARY 2.4. *If X is a stratifiable space and $f: X \to Y$ is an open compact mapping, then Y is a metacompact semi-stratifiable space.*

Proof. The image of a paracompact space under an open compact mapping is metacompact by Theorem 4 of [1]. Since open mappings are pseudo-open, Y is also semi-stratifiable.

If the converse of Theorem 2.3 is true, then another characterization of semi-stratifiable spaces is available. Also, Corollary 2.4 is an analogue of the well-known result that an open compact image of a metric space is a space having a uniform base (metacompact and developable).

3. Mappings from semi-stratifiable and semi-metrizable spaces. Semi-stratifiable and semi-metrizable spaces are closely related in the sense that a first countable semi-stratifiable space is semi-metrizable, and conversely [4, Corollary 1.4]. Creede showed that semi-stratifiable spaces are preserved under closed mappings, but a similar result is not true for semi-metric spaces since there is no guarantee that the image will be first countable, even if the domain is a separable metric. Nor is the property of being semi-metrizable transmitted under an open mapping, for in this case, Creede [5, Theorem 3.4] has exhibited a non-semistratifiable Hausdorff space which is the open image of a separable metric space. However, by placing a suitable restriction on an open mapping, a class of open mappings can be found in which members preserve both semi-stratifiable and semi-metric spaces.

THEOREM 3.1. *If X is semi-stratifiable and $f: X \to Y$ is a pseudo-open finite-to-one mapping, then Y is semi-stratifiable.*

Proof. Let $F \subset Y$ be an arbitrary closed set. Then $f^{-1}(F)$ is closed in X and has a dual semi-stratification $\{U_n\}$. We will use

Lemma 1.4 to show that the correspondence $F \to \{\mathrm{int}[f(U_n)]\}$ is a dual semi-stratification for Y.

Parts (a) and (c) are verified in the same manner as in the proof of Theorem 2.3. To verify (b), assume $z \in \bigcap_{n=1}^{\infty} \mathrm{int}[f(U_n)]$. Then there exist points $x_n \in U_n$ such that $f(x_n) = z$ for every n. Since f is a finite-to-one mapping, there exists an integer m such that $x_m \in \bigcap_{n=1}^{\infty} U_n$. But $\bigcap_{n=1}^{\infty} U_n = f^{-1}(F)$ which implies that $x_m \in f^{-1}(F)$. Hence, $z \in F$ and the proof is complete.

COROLLARY 3.2. *The image of a semi-stratifiable space under an open finite-to-one mapping is semi-stratifiable.*

COROLLARY 3.3. *The image of a semi-metric space under an open finite-to-one mapping is semi-metrizable.*

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