STRATIFIABLE SPACES, SEMI-STRATIFIABLE SPACES, AND THEIR RELATION THROUGH MAPPINGS

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It is shown that the image of a stratifiable space under a pseudo-open compact mapping is semi-stratifiable. By strengthening the mapping from compact to finite-to-one the following results are also obtained. The image of a semi-stratifiable (semi-metric) space under an open finite-to-one mapping is semi-stratifiable (semi-metric).

Notation and terminology will follow that of Dugundji [6]. By a neighborhood of a set A, we will mean an open set containing A, and all mappings will be continuous and surjective.

**Definition 1.1.** A topological space $X$ is a stratifiable space if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^\infty$ of open subsets of $X$ such that

(a) $\bar{U}_n \subset U$,
(b) $U_n^{n=1} = U$,
(c) $U_n \subset V_n$ whenever $U \subset V$.

**Definition 1.2.** A topological space $X$ is a semi-stratifiable space if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^\infty$ of closed subsets of $X$ such that

(a) $U_n^{n=1} = U$,
(b) $U_n \subset V_n$ whenever $U \subset V$.

Ceder [3] introduced $M_s$-spaces and Borges [2] renamed them “stratifiable”, while Creede [4] studied semi-stratifiable spaces. A correspondence $U \rightarrow \{U_n\}_{n=1}^\infty$ is a stratification (semi-stratification) for the space $X$ whenever it satisfies the conditions of Definition 1.1 (1.2).

**Lemma 1.3.** A space $X$ is stratifiable if and only if to each closed subset $F \subset X$ one can assign a sequence $\{U_n\}$ of open subsets of $X$ such that

(a) $F \subset U_n$ for each $n$,
(b) $\cap_{n=1}^\infty \bar{U}_n = F$,
(c) $U_n \subset V_n$ whenever $U \subset V$.

**Lemma 1.4.** A space $X$ is semi-stratifiable if and only if to each closed set $F \subset X$ one can assign a sequence $\{U_n\}$ of open subsets
of $X$ such that
\begin{enumerate}[(a)]
  \item $F \subseteq U_n$ for each $n$, \\
  \item $\bigcap_{n=1}^\infty U_n = F$ \\
  \item $U_n \subseteq V_n$ whenever $U \subseteq V$.
\end{enumerate}

A correspondence $F \to \{U_n\}_{n=1}^\infty$ is a dual stratification (semi-stratification) for the space $X$ whenever it satisfies the three conditions of Lemma 1.3 (1.4). For convenience in the proofs which will be encountered, each member in the range of a correspondence will also be called a dual stratification (semi-stratification) of the closed set to which it is associated.

2. Mappings from stratifiable spaces. We now exhibit a natural way in which semi-stratifiable spaces may arise.

**Definition 2.1.** A mapping $f: X \to Y$ is pseudo-open if for each $y \in Y$ and any neighborhood $U$ of $f^{-1}(y)$, it follows that $y \in \text{int}[f(U)]$.

**Definition 2.2.** A mapping $f: X \to Y$ is compact if $f^{-1}(y)$ is compact for each $y \in Y$.

**Theorem 2.3.** If $X$ is stratifiable and $f: X \to Y$ is a pseudo-open compact mapping, then $Y$ is semi-stratifiable.

**Proof.** Let $F \subseteq Y$ be a closed set. Then $f^{-1}(F)$ is closed in $X$ and, hence, by Lemma 1.3, has a dual stratification $\{U_n\}$. We will show that the correspondence $F \to \{\text{int}[f(U_n)]\}$ is a dual semi-stratification for $Y$ by proving that the collections $\{\text{int}[f(U_n)]\}$ satisfy the requirements of Lemma 1.4.

Part (c) of Lemma 1.4 is easily shown to be satisfied. For if $F$ and $G$ are closed subsets of $Y$ such that $F \subseteq G$, then $f^{-1}(F) \subseteq f^{-1}(G)$, and denoting the dual stratifications of $f^{-1}(F)$ and $f^{-1}(G)$ by $\{U_n\}$ and $\{V_n\}$, respectively, we must have by Lemma 1.3(c) that $U_n \subseteq V_n$ for each $n$. Therefore, $\text{int}[f(U_n)] \subseteq \text{int}[f(V_n)]$.

With regard to part (a), it follows that $F \subseteq \text{int}[f(U_n)]$ for each $n$. This is because each $U_n$ is a neighborhood of $f^{-1}(y)$ for every $y \in F$, and therefore $y \in \text{int}[f(U_n)]$ for every $y \in F$ by hypothesis of $f$ being a pseudo-open mapping.

All that remains to be shown is that $\bigcap_{n=1}^\infty \text{int}[f(U_n)] = F$, and this will verify (b). From the preceding paragraph we know that $F \subseteq \bigcap_{n=1}^\infty \text{int}[f(U_n)]$. To get inclusion in the reverse direction, assume $z \in \bigcap_{n=1}^\infty \text{int}[f(U_n)]$. Then $z \in \text{int}[f(U_n)]$ for every $n$; hence, there exist points $x_n \in U_n$ such that $f(x_n) = z$. Since $f$ is a compact mapping, the sequence $\{x_n\}$ has an accumulation point $x$. Therefore, given any
neighborhood $V$ of $x$, there exist infinitely many integers $n_i$ such that $x_{n_i} \in V$. Thus, $V$ has a nonempty intersection with infinitely many $U_n$, and since we may assume that the collection \{${U_n}$\} is descending, this implies that $V \cap U_n \neq \emptyset$ for every $n$. That is, $x \in \bigcap_{n=1}^{\infty} \overline{U}_n$. But \{${U_n}$\} was a dual stratification for $f^{-1}(F)$ which implies that $\bigcap_{n=1}^{\infty} \overline{U}_n = f^{-1}(F)$. Thus, $x \in f^{-1}(F)$ and $f(x) \in F$. Furthermore, $f(x) = z$ because $x \in \{x_n\}$ and $\{x_n\} \subset f^{-1}(z) = f^{-1}(z)$. Hence, $z \in F$ and the proof is complete.

**Corollary 2.4.** If $X$ is a stratifiable space and $f : X \rightarrow Y$ is an open compact mapping, then $Y$ is a metacompact semi-stratifiable space.

**Proof.** The image of a paracompact space under an open compact mapping is metacompact by Theorem 4 of [1]. Since open mappings are pseudo-open, $Y$ is also semi-stratifiable.

If the converse of Theorem 2.3 is true, then another characterization of semi-stratifiable spaces is available. Also, Corollary 2.4 is an analogue of the well-known result that an open compact image of a metric space is a space having a uniform base (metacompact and developable).

### 3. Mappings from semi-stratifiable and semi-metrizable spaces

Semi-stratifiable and semi-metrizable spaces are closely related in the sense that a first countable semi-stratifiable space is semi-metrizable, and conversely [4, Corollary 1.4]. Creede showed that semi-stratifiable spaces are preserved under closed mappings, but a similar result is not true for semi-metric spaces since there is no guarantee that the image will be first countable, even if the domain is a separable metric. Nor is the property of being semi-metrizable transmitted under an open mapping, for in this case, Creede [5, Theorem 3.4] has exhibited a non-semistratifiable Hausdorff space which is the open image of a separable metric space. However, by placing a suitable restriction on an open mapping, a class of open mappings can be found in which members preserve both semi-stratifiable and semi-metric spaces.

**Theorem 3.1.** If $X$ is semi-stratifiable and $f : X \rightarrow Y$ is a pseudo-open finite-to-one mapping, then $Y$ is semi-stratifiable.

**Proof.** Let $F \subset Y$ be an arbitrary closed set. Then $f^{-1}(F)$ is closed in $X$ and has a dual semi-stratification \{${U_n}$\}. We will use
Lemma 1.4 to show that the correspondence $F \to \{\text{int}[f(U_n)]\}$ is a dual semi-stratification for $Y$.

Parts (a) and (c) are verified in the same manner as in the proof of Theorem 2.3. To verify (b), assume $z \in \bigcap_{n=1}^{\infty} \text{int}[f(U_n)]$. Then there exist points $x_n \in U_n$ such that $f(x_n) = z$ for every $n$. Since $f$ is a finite-to-one mapping, there exists an integer $m$ such that $x_m \in \bigcap_{n=1}^{\infty} U_n$. But $\bigcap_{n=1}^{\infty} U_n = f^{-1}(F)$ which implies that $x_m \in f^{-1}(F)$. Hence, $z \in F$ and the proof is complete.

**COROLLARY 3.2.** The image of a semi-stratifiable space under an open finite-to-one mapping is semi-stratifiable.

**COROLLARY 3.3.** The image of a semi-metric space under an open finite-to-one mapping is semi-metrizable.

**REFERENCES**


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**Texas Christian University**
Mohammad Shafqat Ali and Marvin David Marcus, *On the degree of the minimal polynomial of a commutator operator* ........................................ 561

Howard Anton and William J. Pervin, *Integration on topological semifields* ................................................................. 567

Martin Bartelt, *Multipliers and operator algebras on bounded analytic functions* ................................................................. 575

Donald Earl Bennett, *Aposyndetic properties of unicoherent continua* ............... 585

James W. Bond, *Lie algebras of genus one and genus two* ......................... 591

Mario Borelli, *The cohomology of divisorial varieties* ........................................ 617

Carlos R. Borges, *How to recognize homeomorphisms and isometries* ............. 625

J. C. Breckenridge, *Burkill-Cesari integrals of quasi additive interval functions* ....................................................... 635

J. Csima, *A class of counterexamples on permanents* ........................................ 655

Carl Hanson Fitzgerald, *Conformal mappings onto \( \omega \)-swirly domains* .... 657

Newcomb Greenleaf, *Analytic sheaves on Klein surfaces* ............................. 671

G. Goss and Giovanni Viglino, *C-compact and functionally compact spaces* ............................................................. 677

Charles Lemuel Hagopian, *Arcwise connectivity of semi-aposyndetic plane continua* .......................................................................................... 683

John Harris and Olga Higgins, *Prime generators with parabolic limits* ............. 687

David Michael Henry, *Stratifiable spaces, semi-stratifiable spaces, and their relation through mappings* ........................................ 697

Raymond D. Holmes, *On contractive semigroups of mappings* ...................... 701

Joseph Edmund Kist and P. H. Maserick, *BV-functions on semilattices* ....... 711

Shûichirô Maeda, *On point-free parallelism and Wilcox lattices* ................. 725

Gary L. Musser, *Linear semiprime \((p; q)\) radicals* ................................... 749

William Charles Nemitz and Thomas Paul Whaley, *Varieties of implicative semilattices* .......................................................... 759

Jaroslav Nešetřil, *A congruence theorem for asymmetric trees* ...................... 771

Robert Anthony Nowlan, *A study of H-spaces via left translations* ............... 779

Gert Kjærgaard Pedersen, *Atomic and diffuse functionals on a C*-algebra* .... 795

Tilak Raj Prabhakar, *On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials* .......................... 801

Leland Edward Rogers, *Mutually aposyndetic products of chainable continua* ................................................................. 805

Frederick Stern, *An estimate for Wiener integrals connected with squared error in a Fourier series approximation* .................... 813

Leonard Paul Sternbach, *On k-shrinking and k-boundedly complete basic sequences and quasi-reflexive spaces* ...................... 817

Pak-Ken Wong, *Modular annihilator \( A^*\)-algebras* ................................... 825