A STUDY OF \textit{H}-SPACES VIA LEFT TRANSLATIONS

ROBERT ANTHONY NOWLAN
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$H$-spaces are examined by studying left translations, actions and a homotopy version of left translations to be called homolations. If $(F, m)$ is an $H$-space, the map $s: F \to F^F$ given by $s(x) = L_x$, i.e., $s(x)$ is left translation by $x$, is a homomorphism if and only if $m$ is associative. In general, $s$ is an $A_\infty$-map if and only if $(F, m)$ is an $A_{n+1}$ space.

The action $r: F^F \times F \to F$ is given by $r(\varphi, x) = \varphi(x)$. The map $s$ respects the action only of left translations. In general, $s$ respects the action of homolations up to higher-order homotopies. Each homolation generates a family of maps to be called a homolation family. Denoting the set of all homolation families by $H^\infty(F)$, $s: F \to F^F$ factors through $F \to H^\infty(F)$ and this latter map is a homotopy equivalence.

By a multiplication on a space $F$, we mean a continuous map $m: F \times F \to F$. Let $m$ be a given multiplication on $F$. For any two points $x$ and $y$ of $F$, $m(x, y)$ will be denoted by $xy$ and is called the product of $x$ and $y$. For any point $x$ of $F$, the assignment $x \to yx$ and $x \to xy$ determine respectively the maps

$$L_y: F \to F, \quad R_y: F \to F$$

called the left and right translation of $F$ by $y$.

This paper examines $H$-spaces with strict units by studying left translations and by the introduction of a homotopy version of left translations to be called homolations. One way to use left translations is as follows. If $(F, m)$ is an $H$-space, the map

$$s: F \to F^F$$

given by $s(x) = L_x$, i.e., $s(x)$ is left translation by $x$, is a homomorphism if and only if $m$ is associative. Other properties of $H$-structures on a space $F$ can also be interpreted in terms of properties of the map $s: F \to F^F$.

**DEFINITION 1.** A map $f: F \to Y$ is an $H$-map of the $H$-space $(F, m)$ into the $H$-space $(Y, w)$ if $w \circ (f \times f) \equiv f \circ m$. (We always use "\equiv" to denote "is homotopic to").

In §II we prove that $s$ is an $H$-map if and only if $m$ is homotopy associative. In [2], and [3], Stasheff introduces the concepts of $A_\infty$-spaces and of $A_\infty$-maps, the former generalizes homotopy associativity and the latter generalizes $H$-maps. We will show that $s$
is an $A_\infty$-map if and only if $(F, m)$ is an $A_{\infty+1}$-space.

In §111, $H$-spaces are studied in terms of actions. The action $r: F^F \times F \to F$ is given by $r(\varphi, x) = \varphi(x)$. The cross-section $s: F \to F^F$ respects the action only of left translations. The question arises: of which maps in $F^F$ does $s$ respect the action up to homotopy? This leads to the introduction of $T$-maps, that is maps $f: F \to F$ such that $f \circ m \cong m \circ (f \times 1)$. Such maps resemble left translations. Demanding a closer resemblance leads to the introduction of homolations which are maps $f$ satisfying $f \circ m \cong m \circ (f \times 1)$ up to higher order homotopies.

If $(F, m)$ is an associative $H$-space, a map $w: M \times F \to F$ is a transitive action if $w \circ (1 \times m) = m \circ (w \times 1)$. The action $r: s(F) \times F \to F$, where $s(F)$ is the set of all left translations is an example of a transitive action. A homotopy version of a transitive action is given as follows.

DEFINITION 2. Let $(F, m)$ be an associative $H$-space. A map $w: M \times F \to F$ is a $T$-action if $w \circ (1 \times m) \cong m \circ (w \times 1)$.

If $T(F)$ is the maximal subset of $F^F$ such that $r: T(F) \times F \to F$ is a $T$-action, then $T(F)$ consists of $T$-maps. Generalizing the notions of $T$-actions leads to the concept of $T_\infty$-actions and $T_{\infty+1}$-actions, that is actions $w: M \times F \to F$ satisfying $w \circ (1 \times m) \cong m \circ (w \times 1)$ up to higher order homotopies. It is then shown that a $T_\infty$-action of the set of homolations on $F$ can be given such that $s: F \to F^F$ is a $T_\infty$-map of actions, i.e., $s$ respects the actions of homolations up to higher order homotopies.

Each homolation generates a family of maps to be called a homolation family. Denote by $H_\omega(F)$ the set of all homolation families. In §IV, it is proven that $s: F \to F^F$ factors through $F \to F_\omega(F)$ and that this latter map is a homotopy equivalence.

Throughout this paper, we will be working in the category of $k$-spaces (i.e., compactly generated spaces) as developed in [5]. The reason for this is to allow unlimited use of the “exponential law.” (c.f. Theorem 5, 6 in [5]).

Some of the work included in this paper is contained in my doctoral thesis [1] completed at the University of Notre Dame. Other parts of it were suggested by Professor James D. Stasheff. I deeply appreciate his suggestions and many valuable comments during the writing of this paper.

II. $A_\infty$-maps and $A_{\infty+1}$-spaces We first study $H$-spaces in relation
to cross-sections to evaluation maps. Let $F$ be any space. Let the evaluation map $v: F^e \to F$ be defined by $v(\varphi) = \varphi(e)$, where $\varphi$ is in $F^e$ for some $e$ in $F$. The map $v$ has a cross-section $s: F \to F^e$ if and only if $F$ admits a multiplication with right unit $e$. Given such a cross-section $s$ we can define

$$m(x, y) = s(x)(y)$$

so that $m$ has $e$ as a right unit. Since

$$s(x)(e) = v(s(x)) = x,$$

this multiplication has a two-sided unit if $s$ is a base point preserving map, that is $s(e) = \text{identity}$. We will make this assumption throughout this paper.

If $F$ has a multiplication $m$ with $e$ as right unit, we define $s(x) = L_x$, where $L_x$ is left translation by $x$. It follows that $s$ is a homomorphism if and only if $m$ is associative.

Thus certain properties of $H$-structures on a space $F$ can be interpreted in terms of properties of the map $s: F \to F^e$. As an example we have the following proposition.

**Proposition 1.** The map $s: F \to F^e$ is an $H$-map if and only if $m$ is homotopy associative.

**Proof.** If $s$ is an $H$-map of $(F, m)$ into $(F^e, c)$ (where $c$ is composition of maps), there exists a homotopy

$$G: I \times F^e \to F^e$$

such that

$$G(0, x, y) = c \circ (s \times s)(x, y) = L_x \circ L_y$$

and

$$G(1, x, y) = s \circ m(x, y) = L_{xy}.$$ 

Then $m$ can be shown to be homotopy associative by defining a homotopy

$$G': I \times F^e \to F$$

by

$$(1) \quad G'(t, x, y, z) = G(t, x, y)(z)$$

Conversely, if $m$ is homotopy associative, a homotopy $G'$ exists such that
\[
G'(0, x, y, z) = x(yz)
\]
and
\[
G'(1, x, y, z) = (xy)z
\]
and the homotopy \( G \) can be defined as in (1).

In seeking to generalize this proposition, we first need generalizations of the concepts of homotopy associativity and of \( H \)-map. In [2] and [3], Stasheff introduces the concepts of \( A_n \)-spaces and of \( A_\infty \)-maps; the former generalizes homotopy associativity and the latter generalizes \( H \)-maps. A space which is an \( A_n \)-space for all \( n \) is said to be an \( A_\infty \)-space. Any associative \( H \)-space is an \( A_\infty \)-space. \( A_\infty \)-spaces are homotopy equivalent to associative \( H \)-spaces.

**Definition 3.** An \( A_n \)-structure on a space \( X \) consists of an \( n \)-tuple of maps

\[
X = E_1 \subset E_2 \subset \cdots \subset E_n
\]

\[
\begin{array}{c}
p_1 \\
p_2 \\
p_n
\end{array}
\]

\[
* = B_1 \subset B_2 \subset \cdots \subset B_n
\]

such that \( p_i : \pi_q(E_i, X) \to \pi_q(B_i) \) is an isomorphism for all \( q \), together with a contracting homotopy \( h: CE_{n-1} \to E_n \) of the cone of \( E_{n-1} \), \( CE_{n-1} \) such that \( h(CE_{n-1}) \subset E_i \). Such an \( A_n \)-structure will be denoted by \( (p_1, \ldots, p_n) \). If there exists an infinite collection \( p_1, p_2, \ldots \) such that for each \( n \), \( (p_1, \ldots, p_n) \) is an \( A_n \)-structure, then we call \( (p_1, p_2, \ldots) \) an \( A_\infty \)-structure.

Theorem 5 of [2] asserts that an \( A_n \)-structure on a space \( X \) is equivalent to an "\( A_n \)-form", that is a family of maps \( \{M_i, \ldots, M_n\} \) where each

\[
M_i: I^{i-2} \times X^i \longrightarrow X
\]

is suitably defined on the boundary \( I^{i-1} \) in terms of \( M_j \) for \( j < i \).

**Definition 4.** A space \( X \) together with an \( A_n \)-form will be called an \( A_\infty \)-space.

In this paper, we are more interested in \( A_n \)-forms than \( A_\infty \)-structures, so we introduce the former in some detail. It is first necessary to become acquainted with a special cell-complex \( K_i \) which is homeomorphic to \( I^{i-2} \) for \( i \geq 2 \). The standard cells \( K_i \) are objects
similar to standard simplices $\Delta^i$ and standard cubes $I^i$, having faces and degeneracies. The difference between the $K_i$ and the simplices and the cubes is that:

1. The index $i$ does not refer to the dimension of the cell but rather to the number of factors $X$ with which $K_i$ is to be associated.

2. $K_i$ has degeneracy operators $s_1, \ldots, s_i$ defined on it.

3. $K_i$ has $(i(i - 1)/2) - 1$ faces.

The following description of the indexing of the faces of $K_i$ is due to Stasheff. Consider a word with $i$ letters, and all meaningful ways of inserting one set of parentheses. To each such insertion except for $(x_1, \ldots, x_i)$, there corresponds a cell of $L_i$, the boundary of $K_i$. If the parentheses enclose $x_k$ through $x_{k+s-1}$, we regard this cell as the homeomorphic image of $K_r \times K_s$ ($r + s = i + 1$) under a map which we denote by $\partial_d(r, s)$. Two such cells intersect only on their boundaries and the "edges" so formed correspond to inserting two sets of parentheses in the word. We obtain $K_i$ by induction, starting with $K_2 = \ast$ (a point), supposing $K_2$ through $K_{i-1}$ have been constructed. Then construct $L_i$ by fitting together copies of $K_r \times K_s$ subject to certain conditions given in §2 of [2], that is the fitting together of copies of $K_r \times K_s$ as dictated by the above description of the indexing. Finally, take $K_i$ to be the cone on $L_i$.

The following is part of Theorem 5 of [2].

**Theorem 2.** A space $X$ admits an $A_n$-structure if and only if there exist maps $M_i: K_i \times X^i \rightarrow X$ for $2 \leq i \leq n$ such that

1. $M_i(\ast, e, x) = M_i(\ast, x, e) = x$ for $x$ in $X$, $\ast = K_2$ and

2. For $\rho \in K_r$, $\sigma \in K_s$, $r + s = i + 1$, we have

$$M_i(\partial_d(r, s)(\rho, \sigma), x_1, \ldots, x_i) = M_r(\rho, x, \ldots, x_{k-1}, M_s(\sigma, x_k, \ldots, x_{k+s-1}), \ldots, x_i).$$

We note that an $A_2$-space is just an $H$-space. In the case $i = 3$, $K_3$ is homeomorphic to $I$ and (2) asserts that $M_3$ is a homotopy between $M_3(1 \times 1)$ and $M_3(1 \times K_3)$, to be imprecise between $(xyz)$ and $x(yz)$. Thus $M_3$ is an associating homotopy and $M_2$ is a homotopy associative action.

In the case $i = 4$, we consider the five ways of associating a product of four factors. If the multiplication $M_4$ is a homotopy associative multiplication, the five products are then related by the following string of homotopies:

$$x(y(zw)) \equiv x((yz)w) \equiv (x(yz))w \equiv ((xy)z)w \equiv (xy)(zw) \equiv x(yzw).$$

Thus we have defined a map of $S^1 \times X^4 \rightarrow X$ and the map $M_4$ can
be regarded as an extension of the map to $I^2 \times X^4$.

If $X$ is an associative $H$-space, it admits $A_\infty$-forms; it is only necessary to define

$$M_i(\tau, x_1, \ldots, x_i) = x_1x_2 \cdots x_i$$

for $\tau$ in $K_i$ and $1 \leq i$. This will be called a trivial $A_\infty$-form. If $X$ is an $A_\infty$-space then there is an associative $H$-space $Y$ of the homotopy type of $X$.

**DEFINITION 5.** Let $(X, \{M_i\})$ be an $A_\infty$-space and $(Y, w)$ be an associative $H$-space. A map $f: X \to Y$ is an $A_\infty$-map if there exists maps $h_i: K_{i-1} \times X^i \to Y$, $1 \leq i \leq n$, called sputnik homotopies, such that $h_i = f$ and for $\rho$ in $K_r$, $\sigma$ in $K_s(r + s = i + 1)$, we have

$$h_i(\partial_r(\rho, \sigma), x_1, \ldots, x_i)$$

$$= h_{r-1}(\rho, x_1, \ldots, x_{k+1}, M_i(\sigma, x_k, \ldots, x_{k+s-1}), \ldots, x_i)$$

if $k \neq r$

$$= h_{r-1}(\rho, x_1, \ldots, x_{r-1})h_{s-1}(\sigma, x_r, \ldots, x_i)$$

if $k = r$.

Note that when $n = 2$, $f$ is just an $H$-map, as $h_2$ is a homotopy between $f \circ M_2$ and $w \circ (f \times f)$. In the case $n = 3$, since $K_4$ is homeomorphic to $I^3$, we have a map of $S^1 \times X^3 \to Y$ and $h_3: K_4 \times X^3 \to Y$ can be thought of as an extension of this map to $I^2 \times X^4$.

Consider the following cross-section of $I^2 \times X^4$ showing a typical $I^2$. Assign to the “faces” of $I^2$ the homotopies $h_2 \circ (M_2 \times 1)$, $w \circ (h_2 \times h_1)$, $w \circ (h_1 \times h_2)$, $h_2 \circ (1 \times M_2)$ and $h_1 \circ M_3$ as indicated

The broken line represents a point. The map $h_3$ then appropriately fills in the figure.

A map which is an $A_n$-map for all $n$ will be called an $A_\infty$-map. We are now in a position to prove the following generalization of proposition 1.

**THEOREM 3.** (A) Let $(F, \{M_i\})$ be an $A_n$-space; then $s: F \to F^F$ is an $A_{n-1}$ map.

(B) $s$ can be shown to be an $A_n$-map if and only if $(F, \{M_i\})$ can
be given the structure of an $A_{n+1}$ space.

Proof. (A) Given that $(F, \{M_i\})$ is an $A_n$-space, all that is necessary to show that $s$ is an $A_{n-1}$ map is to define $h_i = s$ and $h_i: K_{i+1} \times F^i \to F^F$ $1 \leq i \leq n - 1$ by

$$h_i(\delta_k(r, t)(\rho, \sigma), x_1, \cdots, x_i) = M_{i+1}(\delta_k(r, t)(\rho, \sigma), x_1, \cdots, x_i, y) .$$

(B) It is clear that $(F, \{M_i\})$ can be extended to an $A_{n+1}$-space (that is there exists a map $M_{n+1}: K_{n+1} \times F^{n+1} \to F$) if and only if there exists a map $h_n: K_{n+1} \times F^n \to F^F$ given by

$$h_n(\delta_k(r, t)(\rho, \sigma), x_1, \cdots, x_n) = M_{n+1}(\delta_k(r, t)(\rho, \sigma), x_1, \cdots, x_n, y) .$$

COROLLARY 4. An $A_\infty$-form on $F$ is equivalent to the existence of sputnik homotopies $h_i: K_{i+1} \times F^i \to F^F$ for all $i$ making $s$ an $A_\infty$-map.

III. $T_\infty$-maps and Homolations. We assume throughout this section that $(F, m)$ is an associative $H$-space with a strict unit. In that case, the map

$$s: F \to F$$

given by

$$s(f)(y) = m(f, y)$$

is a homomorphism.

We now study left translations via actions. The space $F^F$ acts on $F$ by

$$r: F^F \times F \to F$$

$$r(\varphi, f) = \varphi(f) .$$

The cross-section $s$ respects the action only of left translations, for consider the diagram:

$$F^F \times F \xrightarrow{1 \times s} F^F \times F^F$$

$$\downarrow r \quad \quad \quad \quad \downarrow c$$

$$F \xrightarrow{s} F^F .$$

(1)

Suppose

$$s(\varphi(f)) = \varphi \circ s(f) .$$

Since $s$ is left translation, we have $\varphi(\bar{f}y) = \varphi(f)y$, that is the following diagram is commutative.
\[ F \times F \xrightarrow{m} F \]
\[ \varphi \times 1 \downarrow \quad \downarrow \varphi \]
\[ F \times F \xrightarrow{m} F. \]

In particular,

\[ \varphi(y) = \varphi(ey) = \varphi(e)y \]

and \( \varphi \) is left translation by \( \varphi(e) \). So diagram (1) commutes only on \( s(F') \times F' \subset F' \times F' \) where \( s(F') \) is the set of left translations. Thus \( s \) is a map of spaces on which \( s(F') \) acts.

The result tells us something about the action

\[ r: s(F') \times F \longrightarrow F \]

namely, it is transitive.

Note that the following diagram is commutative

\[ s(F') \times F' \times F' \xrightarrow{1 \times m} s(F') \times F' \]
\[ \varphi \times 1 \downarrow \quad \downarrow \varphi \]
\[ F' \times F' \xrightarrow{m} F'. \]

Let us consider the following question: what is the nature of the action \( r \) when diagram (1) is only required to be homotopy commutative. Denote by \( T_2(F') \) the maximal subset of maps \( \varphi \) in \( F' \) such that

\[ s[\varphi(f)] \equiv \varphi_0 \varphi(f) \]

in the sense that there exists a homotopy

\[ \theta_2: I \times T_2(F') \times F \longrightarrow F' \]

such that

\[ \theta_2(0, \varphi, f) = \varphi_0 \varphi(f) \]

and

\[ \theta_2(1, \varphi, f) = s[\varphi(f)]. \]

In this case, it follows that for each \( \varphi \) in \( T_2(F) \) there exists a homotopy

\[ \varphi_2: I \times F^2 \longrightarrow F \]

depending continuously on \( \varphi \) such that
and

\[ \varphi_s(0, f, y) = \varphi(fy) \]

DEFINITION 6. Let \((F, m)\) be an associative \(H\)-space. A map \(f : F \to F\) is a \(T\)-map if there exists a homotopy \(I \times F^2 \to F\) such that \(f \circ m \cong m \circ (f \times 1)\).

Thus we see that the maps in \(T_2(F)\) are \(T\)-maps. The homotopy is given by

\[ \varphi_2(t, f, y) = \theta_2(t, \varphi, f)(y) \]

In particular, we note that for each \(\varphi\) in \(T_2(F)\)

\[ \varphi(y) = \varphi(ey) \cong \varphi(e)y \]

indicating that up to homotopy \(\varphi\) acts like left translation by \(\varphi(e)\). Thus the maps in \(T_2(F)\) in this sense resemble left translations. We will investigate this resemblance further.

Our results show that the action

\[ r : T_2(F) \times F \to F^p \]

is a \(T\)-action in the sense that there exists a homotopy

\[ \lambda_2 : I \times T_2(F) \times F^2 \to F \]

such that

\[ \lambda_2 : r \circ (1 \times m) \cong m \circ (r \times 1) \]

In fact, we can take \(\lambda_2\) to be adjoint to \(\theta_2\):

\[ \lambda_2(t, \varphi, f, y) = \theta_2(t, \varphi, f)(y) \]

If \(\varphi\) is a true left translation, it follows that

\[ \varphi(xyz) = \varphi(xy)z = \varphi(x)yz \]

however for a map \(\varphi\) in \(T_2(F)\), the most we can claim using a rather loose notation is that:

\[ \varphi(xyz) \cong \varphi(xy)z \cong \varphi(x)yz \cong \varphi(xyz) \]

This string of homotopies defines a map

\[ \hat{I}^2 \times F \to F \]

where \(\hat{I}^2\) is the boundary of \(I^2\).
This can be illustrated in the following diagram, representing \( \hat{I}^2 \times F^3 \) showing only \( \hat{I}^2 \) with "faces" labeled by the homotopies connecting the maps given above. Note that the edge of \( \hat{I}^2 \) represented by the broken line is just a point. (This is because \( F \) is an associative \( H \)-space. If \( F \) were only homotopy associative, this face would be labeled by the associating homotopy applied to \( \varphi(x), y, z \). The following discussion could be carried out for \( A_n \)-spaces but the details are bad enough in the associative case, which is the case of interest for applications [1].)

![Diagram](image)

The problem of making a map \( \varphi \) in \( T_2(F') \) more closely "resemble" a left translation, requires that we be able to extend the map

\[
\hat{I}^2 \times F^3 \rightarrow F
\]

to a map

\[
I^2 \times F^3 \rightarrow F.
\]

Thus we will need higher homotopy conditions on the maps \( \varphi \) in \( T_2(F') \). Suppose for the moment that there exists a map

\[
\varphi_3: I^2 \times F^3 \rightarrow F
\]

such that

\[
\varphi_3(0, t_2, x, y, z) = \varphi_3(t_2, xy, z) \\
\varphi_3(t_1, 0, x, y, z) = \varphi_3(t_1, x, yz) \\
\varphi_3(1, t_2, x, y, z) = \varphi(x)yz \\
\varphi_3(t_1, 1, x, y, z) = \varphi_3(t_1, x, y)z.
\]

Let \( T_3(F) \) denote the maximal subset of \( T_2(F') \) such that for each \( \varphi \) in \( T_3(F) \), there exists \( \varphi_3 \) and \( \varphi_2 \) depending continuously on \( \varphi \) and \( \varphi_2 \) subject to the conditions already mentioned. In this case, the action \( r: T_3(F) \times F \rightarrow F \) is such that there exist maps
\[ \lambda_2: I \times T_3(F) \times F^2 \longrightarrow F \]
such that
\[ \lambda_2: r(1 \times m) \cong m(r \times 1) \]
and
\[ \lambda_3: I^2 \times T_3(F) \times F^2 \longrightarrow F \]
such that
\[ \lambda_3(0, t_2, \varphi, x, y, z) = \lambda_2(t_2, \varphi, xy, z) \]
\[ \lambda_3(t_1, 0, \varphi, x, y, z) = \lambda_2(t_1, \varphi, x, yz) \]
\[ \lambda_3(1, t_2, \varphi, x, y, z) = r(\varphi, x) \cdot yz \]
and
\[ \lambda_3(t_1, 1, \varphi, x, y, z) = \lambda_2(t_1, \varphi, x, y) \cdot z . \]

This latter map is given by
\[ \lambda_3(t_1, t_2, \varphi, x, y, z) = \lambda_2(t_1, \varphi, x, y, z) . \]

On the other hand, there exist maps
\[ \theta_2: I \times T_3(F) \times F \longrightarrow F^v \]
such that
\[ \theta_2: \varphi \circ s(f) \cong s[\varphi(f)] \]
and
\[ \theta_3: I^2 \times T_3(F) \times F^{v_2} \longrightarrow F^v \]
such that
\[ \theta_3: (t_1, t_2, \varphi, x, y)(z) = \lambda_3(t_1, t_2, \varphi, x, y, z) . \]

Parallel to every demand that a map \( \varphi: F \rightarrow F \) more closely resemble a left translation by satisfying higher homotopy conditions will be the requirement of higher homotopy conditions on the action \( r \) and similar higher homotopy conditions on the map \( s \).

**Definition 7.** Let \((X, m)\) be an associative \(H\)-space. A map \( \varphi: X \rightarrow X \) is a \(T_n\)-map of \(X\) into itself if there exists a family of maps
\[ \varphi_i: I^{i-1} \times X^i \longrightarrow X \quad 1 \leq i \leq n \]
such that \( \varphi_1 = \varphi \) and
In case \( \phi_t \) exists for all \( i \), we call \( \varphi \) a homolation, that is, a homotopy translation. Denote the set of all homolations by \( T_\infty(F) \).

**Definition 8.** Let \((F, m)\) be an associative \( H \)-space. A homolation family on \( F \) is a collection of maps \( \{ \varphi_i : I^{i-1} \times F^i \to F, \forall i \geq 1 \} \) where \( \varphi_1 \) is a homolation and \( \varphi_1 : F \to F \) is a homotopy equivalence. We will denote by \( H^\infty(F) \), the set of all homolation families. \( H^\infty(F) \) is a subspace of \( C(F; F) \times C(I \times F^{i-1}; F) \times \cdots \) where \( C(I^j \times F^{i+j}; F) \) is the set of all continuous maps \( f : I^j \times F^{i+j} \to F \) (with the \( k \)-topology derived from the compact-open topology).

**Definition 9.** Let \((X, m)\) be an associative \( H \)-space. A map 
\[ w : M \times X \to X \]
of \( M \) on \( X \) is said to be a \( T_n \)-action if there exist maps
\[ w_i : I^{i-1} \times M \times X^i \to X \quad 1 \leq i \leq n \]
such that \( w_1 = w \) and
\[ w_i(t, \cdots, t_{i-1}, g, x_1, \cdots, x_i) = w_{i-1}(t, \cdots, t_{i-1}, g, \cdots, x_k x_{k+1}, \cdots, x_i) \quad \text{if } t_k = 0 \]
\[ = w_k(t, \cdots, t_{k-1}, g, x_1, \cdots, x_k)(x_{k+1} x_{k+2} \cdots x_i) \quad \text{if } t_k = 1. \]
If a map \( w : M \times X \to X \) is a \( T_n \)-action for all \( n \), then \( w \) is said to be a \( T_\infty \)-action.

**Theorem 5.** Let \( T_n(F) \) denote the maximal subset of \( F^\infty \) such that there exist maps \( \lambda_i : I^{i-1} \times T_n(F) \times F^i \to F \) for \( 1 \leq i \leq n \) making \( r : T_n(F) \times F \to F \) a \( T_n \)-action; then \( T_n(F) \) consists of \( T_n \)-maps.

**Proof.** We may define the maps
\[ \varphi_i : I^{i-1} \times F^i \to F \quad 1 \leq i \leq n \]
by
\[ \varphi_i(t_1, \cdots, t_{i-1}, f_1, \cdots, f_i) = \lambda_i(t_1, \cdots, t_{i-1}, \varphi, f_1, \cdots, f_i). \]

**Definition 10.** Let \((X, m)\) and \((M, v)\) be associative \( H \)-spaces and \( w : M \times X \to X \) be a \( T_n \)-action. A homomorphism \( f : X \to M \) is said to be a \( T_n \)-map of actions if there exist maps
\[ \varphi_i : I^{i-1} \times F^i \to F \quad 1 \leq i \leq n \]
such that $\theta_i = 1_M$ and

$$
\begin{align*}
\theta_i(t_1, \ldots, t_{i-1}, g, x_1, \ldots, x_{i-1}) \\
= \theta_{i-1}(t_1, \ldots, \hat{t}_k, x_k, \ldots, x_{i-1} + x_k, x_{k+1}, \ldots, x_{i-1}) \\
& \quad \text{if } t_k = 0, k \neq i - 1 \\
= v[\theta_{i-1}(t_1, \ldots, t_{i-2}, g, x_1, \ldots, x_{i-2})] f(x_{i-1}) \\
& \quad \text{if } t_{i-1} = 0 \\
= f[m(w_k(t_1, \ldots, t_{k-1}, g, x_1, \ldots, x_k), x_{k+1}x_{k+2} \ldots x_i)] \\
& \quad \text{if } t_k = 1.
\end{align*}
$$

If $\theta_i$ exists for all $i$, then $f$ is said to be a $T_\infty$-map of actions.

**Corollary 6.** The map $r: T_\infty(F) \times F$ is a $T_\infty$-action and $s$ is then a $T_\infty$-map of actions.

**Proof.** Define $\lambda_i: I^{i-1} \times T_\infty(F) \times F \rightarrow F$ by

$$
\lambda_i(t_1, \ldots, t_{i-1}, \varphi, f_1, \ldots, f_i) = \varphi_i(t_1, \ldots, t_{i-1}, f_1, \ldots, f_i)
$$

and

$$
\theta_i: I^{i-1} \times T_\infty(F) \times F \rightarrow F^F
$$

by

$$
\theta_i(t_1, \ldots, t_{i-1}, \varphi, f_1, \ldots, f_{i-1})(f_i) = \lambda_i(t_1, \ldots, t_{i-1}, \varphi, f_1, \ldots, f_i).
$$

**IV.** The homotopy equivalence of $F$ and $H_\infty(F)$. As we have seen, we can identify an associative $H$-space with the set of left translations of that space. We note that this identification of $F$ in $F^F$ as left translation is not homotopy invariant: $\varphi(f \cdot x) = \varphi(f)x$ is not a homotopy statement. Our definition of homolation is homotopy invariant and it characterizes $F \rightarrow F^F$ from a homotopy point of view.

We are now in a position to prove the following theorem. Recall that $H_\infty(F)$ is the set of all homolation families.

**Theorem 7.** If $(F, m)$ is a connected associative $H$-space, the map $s: F \rightarrow F^F$ factors through $H_\infty(F)$, and the factor $F \rightarrow H_\infty(F)$ is a homotopy equivalence.

**Proof.** Define a map

$$
\tau: F \rightarrow H_\infty(F)
$$

as follows:
where
\[ \varphi_F : F \to F \]
is given by
\[ \varphi_F(g) = fg \]
that is left translation of \( F \). \( \varphi_F \) is a homotopy equivalence since \( F \) is connected (see [4]).

The remaining maps are given by
\[ \varphi_k(t_1, \ldots, t_{k-1}, f_1, \ldots, f_k) = ff_1 \cdots f_k \quad \text{for all } k. \]
The map \( \tau \) is continuous, since the composition of maps
\[ F' \to C(F; F) \times C(I \times F^2; F) \cdots \overset{p^{(k)}}{\longrightarrow} C(I^{k-1} \times F^k; F) \]
is continuous for each \( k \) and \( p^{(k)} \) is projection onto the corresponding factor.

On the other hand, define the map
\[ \mu : H^\infty(F) \to F \]
by
\[ \mu(I) = \gamma_1(e) \]
where \( I = \{\gamma_1, \gamma_2, \ldots\} \) is in \( H^\infty(F) \) and \( e \) is the unit of \( F \).

The map \( \mu \) is continuous, since it is the composition of maps
\[ H^\infty(F) \overset{p_1}{\longrightarrow} H^\infty(F)_1 = T_\infty(F) \overset{w_e}{\longrightarrow} F \]
where \( p_1 \) is projection of \( H^\infty(F) \) on that part of \( H^\infty(F) \) contained in \( F^p \), namely the set of homotations, here denoted by \( H^\infty(F)_1 \), and the map \( w_e \) is the evaluation map at \( e \) (continuous in the \( k \)-topology).

Note that \( \mu(\tau(f)) = \mu(\Phi_f) = \varphi_f(e) = fe = f \) so that \( \mu \circ \tau = 1_F \).

On the other hand
\[ \tau \circ \mu(I) = \tau(\gamma_1(e)) = \Phi_{\gamma_1(e)} = \{\varphi_{\gamma_1(e)}^{(1)}, \varphi_{\gamma_1(e)}^{(2)} \cdots\}. \]

We claim that \( \tau \circ \mu \equiv 1_{H^\infty(F)} \), that is there exists a map
\[ H_1 : H^\infty(F) \longrightarrow H^\infty(F) \]
such that \( H_0 = 1_{H^\infty(F)} \) and \( H_1 = \tau \circ \mu \).

To see this, let \( H^\infty(F)_k \) be the subspace of \( H^\infty(F) \) which is contained in \( C(I^{k-1} \times F^k; F) \). The map \( H_i = \{H_i^1, H_i^2, \ldots\} \) will consist of homotopies
Define $H^k_i: \mathcal{H}^\infty(F) \to \mathcal{H}^\infty(F)_k$ for each $k$ such that $H^0_k = 1_{\mathcal{H}^\infty(F)_k}$ and $H^k_1 = \tau \circ \mu | \mathcal{H}^\infty(F)_k$ and the $H^k_i$ are compatible.

The map is continuous as each $\gamma_{k+1}$ in $\Gamma$ is continuous and $\Gamma \to \gamma_{k+1}$ is continuous being projection.

Note if $t_j = 0$

$$H^k_i(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) = \gamma_k(t, t_1, \cdots, t_{k-1}, e, f_1, \cdots, f_k)$$

while if $t_j = 1$

$$H^k_i(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) = \gamma_{j+1}(t, t_1, \cdots, t_{j-1}, e, f_1, \cdots, f_j)(f_{j+1}, \cdots, f_k)$$

Thus $\{H^k_i\}$ is in $\mathcal{H}^\infty(F)$. Further

$$H^k_0(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) = \gamma_k(t, \cdots, t_{k-1}, e, f_1, \cdots, f_k) = \gamma_k(t, \cdots, t_{k-1}, f_1, \cdots, f_k).$$

Thus $H^k_0 = 1_{\mathcal{H}^\infty(F)_k}$ $\{H^k_i(\Gamma)\} = \Gamma$ and

$$H^k_1(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) = \gamma_1(e)f_1 \cdots f_k = \varphi_1^1(e)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k)$$

Thus $H^k_1 = \tau \circ \mu | \mathcal{H}^\infty(F)_k$, $\{H^k_i(\Gamma)\} = \tau \circ \mu(\Gamma)$. This completes the proof that $F$ and $\mathcal{H}^\infty(F)$ are homotopy equivalent.

Now $\mathcal{H}^\infty(F)$ is itself an $\mathcal{H}$-space; we can define composition of families as well as just maps $F \to F$ (see [1]). The map $F \to \mathcal{H}^\infty(F)$ is an $\mathcal{A}_\infty$-map and hence induces $B_F \to B_{\mathcal{H}^\infty(F)}$ which is again a homotopy equivalence if $F$ is a $CW$-complex.

In my thesis [1], I show that $B_{\mathcal{H}^\infty(F)}$ is a classifying space for fibrations with $\mathcal{A}_\infty$-actions of $F$ on the total space. The above homotopy equivalence then shows a fibre space admits such an $\mathcal{A}_\infty$-action if and only if it admits an associative action.
REFERENCES


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