ATOMIC AND DIFFUSE FUNCTIONALS ON A $C^*$-ALGEBRA

GERT KJÆRGAARD PEDERSEN
ATOMIC AND DIFFUSE FUNCTIONALS
ON A C*-ALGEBRA

G. K. Pedersen

It is shown that the notion and basic properties of atomic and diffuse measures have exact analogues in the theory of functionals on operator algebras.

We regard a C*-algebra $A$ as the non-commutative analogue of an algebra $C_0(T)$ of continuous functions vanishing at infinity on some locally compact space $T$. It has been shown in [3], [4], [5], [13] and [14] that, at least when $A$ is separable, there is also a reasonable analogue of the Borel functions on $T$, namely the $\sigma$-closure $\mathcal{B}_A$ of $A$. In this paper we prove that $\mathcal{B}_A$ has an abundance of minimal projections, corresponding to points in $T$, and thus the notion of atomic and diffuse measures on $T$ can be generalized to the non-commutative situation, since the diffuse measures are characterized as those measures that vanish at all points of $T$.

Let $A$ be a separable C*-algebra and denote by $P$ the set of pure states of $A$. Choose in $P$ a maximal set \{f_t: t \in T\} of pairwise inequivalent pure states of $A$. If $(\pi_t, H_t)$ denotes the irreducible representation of $A$ corresponding to $f_t$, we define the reduced atomic representation $\rho$ of $A$ as operators on the Hilbert space $H_a = \sum H_t$ by

$$\rho(x)(\sum \xi_t) = \sum \pi_t(x)\xi_t.$$  

The reduced atomic representation is faithful and each pure state of $A$ is a vector functional from $H_a$. Since any other choice of a maximal set in $P$ will give an equivalent representation, the reduced atomic representation is essentially unique. In particular the cardinality of $T$ is uniquely determined as the cardinality of the set $\hat{A}$ of equivalence classes of irreducible representations of $A$. In what follows we shall identify $A$ with its image $\rho(A)$.

Let $\mathcal{B}_A$ denote the monotone $\sigma$-closure of the self-adjoint part of $A$. Then $\mathcal{B}_A = (\mathcal{B}_A^+ + i\mathcal{B}_A^-)$ is a C*-algebra in $B(H)$ called the Baire operators of $A$ [14, Theorem 1]. Each representation $(\pi, H)$ of $A$ extends to a $\sigma$-normal representation of $\mathcal{B}_A$ ([3, Theorem 3.2]) such that $\pi(\mathcal{B}_A)$ is the monotone $\sigma$-closure of $\pi(A)$ in $B(H)$ ([13, Proposition 4.2]). In particular, if $(\pi, H)$ is irreducible we have $\pi(\mathcal{B}_A) = B(H)$.

**Theorem 1.** There is a bijective correspondence between pure states of $A$ and minimal projections of $\mathcal{B}_A$. 

795
Proof. Since \( A \) is separable, each point \( f \) in \( P \) is a closed \( G_\delta \) set. Hence there is a peaking element \( x \) in \( A^+ \), with \( \|x\| = 1 \), such that \( f(x) = 1 \) and \( g(x) < 1 \) for each state \( g \neq f \) ([7, Theorem 9]). We have \( x^* \perp p \), where \( p \) is a projection in \( \mathcal{B}_A \). If \( \xi \) is a unit vector in \( H \), representing \( f \), then \( (p\xi|\xi) = 1 \). For any unit vector \( \eta \) in \( H \), which is not a multiple of \( \xi \) we have \( (y\eta|\eta) \neq (y\xi|\xi) \) for some \( y \) in \( A \); hence \( (p\eta|\eta) < 1 \). It follows that \( p \) is the one-dimensional projection on the subspace spanned by \( \xi \), and consequently minimal.

If, conversely, \( p \) is a minimal projection in \( \mathcal{B}_A \), then \( p\mathcal{B}_A p \) is a commutative algebra, isomorphic with the complex field. The functional \( f \) on \( \mathcal{B}_A \), defined by \( f(x) = pxp \), is the unique state extension of the identity map on \( p\mathcal{B}_A p \) ([11, Theorem 1.2]), which is pure; and therefore \( f \) is a \( \sigma \)-normal pure state of \( \mathcal{B}_A \). But then \( f \in P \). If \( \xi \) is a unit vector in \( pH \), then \( f(x) = (x\xi|\xi) \) and it follows from the first part of the proof that \( p \) is one-dimensional. Thus the correspondence is bijective, and the theorem follows.

Corollary 2. There is a bijective correspondence between elements in \( \hat{A} \) and minimal projections in the center \( \mathcal{C} \) of \( \mathcal{B}_A \).

Remark. Since \( \mathcal{B}_A \) has a unit, we can identify \( \mathcal{C} \) with a \( \sigma \)-closed algebra of bounded functions on \( \hat{A} \). The projections in \( \mathcal{C} \) then constitute the sets in a \( \sigma \)-field on \( \hat{A} \), called the Davies-Borel structure on \( \hat{A} \). The above corollary tells us that points in \( \hat{A} \) are Davies-Borel sets (cf. [5, Theorem 2.9]).

Let \( \mathcal{F} \) denote the smallest monotone closed \( C^* \)-subalgebra of \( \mathcal{B}_A \), which contains all minimal projections of \( \mathcal{B}_A \). Then \( \mathcal{F} \) can be indentified with the set of operators \( x \) in the direct sum \( \sum_{t \in T} B(H_t) \), such that \( x_t = 0 \) except for countably many \( t \) in \( T \). In particular, \( \mathcal{F} \) is an ideal of \( \mathcal{B}_A \).

Definition. A positive functional \( f \) on \( A \) is called atomic if there is a projection \( p \) in \( \mathcal{F} \) such that \( f(1 - p) = 0 \); \( f \) is called diffuse if it vanishes at all minimal projections of \( \mathcal{B}_A \).

Proposition 3. Each positive functional \( f \) on \( A \) has a unique decomposition \( f = f_a + f_d \) such that \( f_a \) is atomic and \( f_d \) is diffuse. Moreover, \( f_a \) and \( f_d \) are centrally orthogonal.

Proof. Let \( \alpha \) be the norm of the functional \( f|\mathcal{F} \) on \( \mathcal{F} \). There is then a sequence \( \{p_n\} \) in the unit ball of \( \mathcal{F}^+ \) such that \( f(p_n) \rightarrow \alpha \).
Replacing \( p_n \) with its range projection, we may assume that all \( p_n \) are projections. Let \( p \) be the central support of \( \sqrt{p_n} \). Then
and \( f(p) = a \). Put \( f_a(x) = f(px) \) and \( f_d(x) = f((1 - p)x) \). Then \( f_a(1 - p) = 0 \), hence \( f_a \) is atomic; and for each \( x \) in \( \mathcal{F}^+ \), with \( \|x\| \leq 1 \), we have \( f(x(1 - p) + p) \leq \alpha \), hence \( f_d(x) = 0 \), and thus \( f_d \) is diffuse. By construction \( f_a \) and \( f_d \) are centrally orthogonal.

**Remark.** We see from the proof that a bounded functional will be atomic (respectively diffuse) if and only if the restriction to \( \mathcal{C} \) induces an atomic (respectively diffuse) measure on the Davies-Borel structure of \( \hat{A} \).

**Proposition 4.** A positive functional \( f \) on \( A \) is atomic exactly if it has the form \( f = \Sigma a_n f_n \), with \( f_n \) in \( P \). Moreover, the summands can be chosen such that \( f_n \perp f_m \) for \( n \neq m \).

**Proof.** If \( f \) is atomic and \( f(1 - p) = 0 \) for a projection \( p \) in \( \mathcal{F} \), then, assuming that \( p \in \mathcal{C} \), we have \( p = \Sigma p_k \), where each \( p_k \) is a minimal projection in \( \mathcal{C} \). Thus \( f = \Sigma f_k \), where \( f_k(x) = f(p_k x) \), and each \( f_k \) is a \( \sigma \)-normal functional on \( B(H_k) = \mathcal{P}(\mathcal{F}_k) \). There is then for each \( k \) an orthonormal basis \( \{\xi_{nk}\} \) for \( H_k \) and a sequence \( \{\alpha_{nk}\} \) of positive constants such that \( f_k(x) = \Sigma \alpha_{nk} \langle x \xi_{nk} | \xi_{nk} \rangle \), for all \( x \) in \( \mathcal{F}_k \). If \( f_{nk} \) denotes the pure state of \( A \) determined by \( \xi_{nk} \), then \( f = \Sigma \alpha_{nk} f_{nk} \), and since the \( f_{nk} \)'s are supported by pairwise orthogonal (minimal) projections in \( \mathcal{F}_k \), they are themselves orthogonal.

Conversely, if \( f = \Sigma \alpha_n f_n \), with all \( f_n \) in \( P \), then for each \( n \) let \( p_n \) be the minimal projection in \( \mathcal{F}_n \) such that \( f_n(p_n) = 1 \). Then \( p = \vee p_n \in \mathcal{F} \) and \( f(1 - p) = 0 \). Hence \( f \) is atomic, completing the proof.

**Definition.** An atom for a positive functional \( f \) on \( A \) is a projection \( p \) in \( \mathcal{F}_A \) such that \( f(p) > 0 \), but \( f(q)f(p - q) = 0 \), for any projection \( q \) in \( \mathcal{F}_A \) smaller than \( p \).

**Proposition 5.** A positive functional is diffuse exactly if it has no atoms.

**Proof.** Assume that \( p \) is an atom for \( f \). Then the state \( g \) of \( A \) given by \( g(x) = f(p)^{-1} f(px p) \) is multiplicative, hence pure, on \( p \mathcal{P}_A p \). Since \( g \) is the unique state extension from \( p \mathcal{P}_A p \) to \( \mathcal{P}_A \) ([11, Theorem 1.2]), we conclude that \( g \in P \). There is then a minimal projection \( q \) in \( \mathcal{F}_A \) such that \( g(q) = 1 \). Since \( pqp \in \mathcal{F}, f \) is not diffuse.

Conversely, if \( f = f_a + f_d \), with \( f_a \neq 0 \), then from Proposition 4 there is a minimal projection \( p \) in \( \mathcal{F}_A \) such that \( f_a(p) > 0 \). Clearly \( p \) is an atom for \( f \), completing the proof.
The following proposition generalizes a well-known theorem from measure theory.

**Proposition 6.** If \( f \) is a diffuse functional on \( A \) then, corresponding to each projection \( p \) in \( \mathcal{B}_A \) and each positive \( \alpha < f(p) \), there is a projection \( q \) in \( \mathcal{B}_A \), with \( q \leq p \) and \( f(q) = \alpha \).

**Proof.** Since \( p \) is not an atom for \( f \), there is a projection \( p_0 < p \) such that \( 0 < f(p_0) < f(p) \). Then either \( f(p_\bot) \leq f(p) \) or \( f(p - p_0) \leq \frac{1}{2} f(p) \). Repeating this procedure we see that for any \( \varepsilon > 0 \) there is a projection \( q_0 \leq p \) such that \( 0 < f(q_0) < \varepsilon \).

Now let \( (\pi_f, H_f) \) be the representation of \( A \) corresponding to \( f \), and let \( \xi_f \) be a vector in \( H_f \) such that \( f(x) = (\pi_f(x)\xi_f, \xi_f) \), for all \( x \) in \( \mathcal{B}_A \). Let \( \{p_i\} \) be a maximal family of nonzero, orthogonal projections in \( \pi_f(\mathcal{B}_A) \) such that \( \Sigma p_i \leq \pi_f(p) \), and \( \Sigma (p_i\xi_f, \xi_f) \leq \alpha \) for all finite sums. Since \( \pi_f(\mathcal{B}_A) \) is a von Neumann algebra ([8, Theorem 2]), \( p_0 = \Sigma p_i \in \pi_f(\mathcal{B}_A) \). It follows from spectral theory that there is a projection \( q \) in \( \mathcal{B}_A \), with \( q \leq p \) and \( \pi_f(q) = p \).

If we had \( f(q) < \alpha \), then from the above we could find a projection \( q_0 \leq p - q \), such that \( 0 < f(q_0) < \alpha - f(q) \). But then \( \pi_f(q_0) \) could be adjoined to the family \( \{p_i\} \); a contradiction. Therefore \( f(q) = \alpha \), completing the proof.

For the sake of convenience we have stated all theorems in terms of bounded functionals. However, it is quite easy to extend the results to a large and important class of unbounded functionals.

Let \( f \) be an extended valued, positive, \( \sigma \)-normal functional on \( \mathcal{B}_A^+ \) which is majorized by an invariant convex functional \( \rho \) on \( \mathcal{B}_A^+ \) (see [13, §2] for definition). Assume furthermore that there is a sequence \( \{e_n\} \) in \( \mathcal{B}_A^+ \) such that \( \Sigma e_n = 1 \), and \( \rho(e_n) < \infty \) for each \( n \). These conditions are satisfied if \( f \) is a \( \sigma \)-finite \( \sigma \)-trace on \( \mathcal{B}_A \) (4, 5) – take \( \rho = f \) or \( f \) is a \( C^* \)-integral of \( A \) ([1, Proposition 4.4] and [13, Theorem 2.5]).

For each \( x \) in \( \mathcal{B}_A \) and each \( n \), the element \( e_n x \) belongs to the set of definition for \( f \), and
\[
|f(e_n x)|^2 \leq f(e_n) f(x^* e_n x) \leq \rho(e_n) \rho(x^* e_n x) \leq \|x\|^2 \rho(e_n)^2.
\]

If \( f_n(x) = f(e_n x) \), then \( \{f_n\} \) is a sequence of \( \rho \)-normal bounded functionals on \( \mathcal{B}_A \) such that \( f(x) = \Sigma f_n(x) \) for all \( x \) in \( \mathcal{B}_A^+ \). For each \( n \) there is a central projection \( p_n \) in \( \mathcal{F} \) such that \( f_n(p_n \cdot) \) is atomic and \( f_n((1 - p_n) \cdot) \) is diffuse. Using \( p = \sqrt{p_n} \) (\( \in \mathcal{C} \cap \mathcal{F} \)) we see that Proposition 3 is valid for \( f \).

To show that most of Proposition 4 holds also for an unbounded atomic functional \( f \) of the above type, we notice that, as in the
proof of Proposition 4, we can write $f = \Sigma f_k$, where each $f_k$ is a $\sigma$-normal (unbounded) functional on an algebra $B(H_k)$. If $p$ is a one-dimensional projection in $H_k$, then $e_n p \neq 0$ for some $n$ and therefore $p = \|pe_n p\|^{-1} pe_n p$. It follows that

$$
\rho(p) = \|pe_n p\|^{-1} \rho\left(\frac{1}{n} pe_n\right) \leq \|pe_n p\|^{-1} \rho(e_n) < \infty.
$$

This proves that $f_k$ on $B(H_k)$ is majorized by an invariant convex functional, which is finite at each operator in $B(H_k)$ of finite rank. Thus $f_k$ is a $C^*$-integral on the $C^*$-algebra of compact operators on $H_k$, and from [10, Theorem 3.8] there is a $b_k$ in $B(H_k)^+$ such that $f_k(x) = \text{tr}(b_k x)$, for all $x$ in $B(H_k)^+$. Hence $f_k$ (and $f$) can be expressed as a sum $\Sigma a_{nk} f_{nk}$ with $f_{nk}$ in $P$ for all $n$ (and $k$). If each $b_k$ can be diagonalized in $B(H_k)$ then $f$ can be written as a weighted sum of mutually orthogonal pure states. This is trivially the case if $f$ is a trace, since then each $f$ is a multiple of $\text{tr}$. In general $b_k$ can not be diagonalized, hence a decomposition in mutually orthogonal bounded functionals is not possible.

Proposition 5 and 6 can be generalized with the same ease. We leave the details to the reader.

Finally we notice that the condition of separability for the $C^*$-algebra $A$ is used primarily to ensure that $\mathscr{D}$ is "large enough" under irreducible representations of $A$. When $A$ is nonseparable there may be irreducible representations of $A$ on nonseparable Hilbertspaces. The pure states of $A$ will then correspond to minimal projections in the Jordan algebra of Borel operators of $A$ defined in [2, § 2.4]. This provides a method for studying atomic functionals on nonseparable $C^*$-algebras. Another way is to define an atomic functional on $A$ as one which is supported on a sequence of minimal projections from the enveloping von Neumann algebra of $A$. But in this case the relation to measure theory becomes less clear.

References

8. R. V. Kadison, Operator algebras with a faithful weakly-closed representation,


Received May 4, 1970

University of Copenhagen, Denmark
The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
Mohammad Shafqat Ali and Marvin David Marcus, *On the degree of the minimal polynomial of a commutator operator* ........................................ 561
Howard Anton and William J. Pervin, *Integration on topological semifields* ................................................................. 567
Martin Bartelt, *Multipliers and operator algebras on bounded analytic functions* ........................................................ 575
Donald Earl Bennett, *Aposyndetic properties of unicoherent continua* ........ 585
James W. Bond, *Lie algebras of genus one and genus two* ....................... 591
Mario Borelli, *The cohomology of divisorial varieties* ........................................ 617
Carlos R. Borges, *How to recognize homeomorphisms and isometries* .......... 625
J. C. Breckenridge, *Burkill-Cesari integrals of quasi additive interval functions* .......................................................... 635
J. Csima, *A class of counterexamples on permanents* ................................. 655
Carl Hanson Fitzgerald, *Conformal mappings onto ω-swirly domains* .......... 657
Newcomb Greenleaf, *Analytic sheaves on Klein surfaces* .............................. 671
G. Goss and Giovanni Viglino, *C-compact and functionally compact spaces* .......................................................... 677
Charles Lemuel Hagopian, *Arcwise connectivity of semi-aposyndetic plane continua* .......................................................... 683
John Harris and Olga Higgins, *Prime generators with parabolic limits* ......... 687
David Michael Henry, *Stratifiable spaces, semi-stratifiable spaces, and their relation through mappings* .......................................... 697
Raymond D. Holmes, *On contractive semigroups of mappings* ................. 701
Joseph Edmund Kist and P. H. Maserick, *BV-functions on semilattices* .... 711
Shûichirô Maeda, *On point-free parallelism and Wilcox lattices* ................. 725
Gary L. Musser, *Linear semiprime (p; q) radicals* ....................................... 749
William Charles Nemitz and Thomas Paul Whaley, *Varieties of implicitive semilattices* ........................................................ 759
Jaroslav Nešetřil, *A congruence theorem for asymmetric trees* ....................... 771
Gert Kjærgaard Pedersen, *Atomic and diffuse functionals on a C*-algebra* .... 795
Tilak Raj Prabhakar, *On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials* .......................................................... 801
Leland Edward Rogers, *Mutually aposyndetic products of chainable continua* ........................................................ 805
Frederick Stern, *An estimate for Wiener integrals connected with squared error in a Fourier series approximation* ......................... 813
Leonard Paul Sternbach, *On k-shrinking and k-boundedly complete basic sequences and quasi-reflexive spaces* ...................... 817
Pak-Ken Wong, *Modular annihilator A*-algebras ........................................ 825