ATOMIC AND DIFFUSE FUNCTIONALS ON A C*-ALGEBRA

GERT KJÆRGAARD PEDERSEN
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G. K. Pedersen

It is shown that the notion and basic properties of atomic and diffuse measures have exact analogues in the theory of functionals on operator algebras.

We regard a C*-algebra A as the non-commutative analogue of an algebra $C_0(T)$ of continuous functions vanishing at infinity on some locally compact space T. It has been shown in [3], [4], [5], [13] and [14] that, at least when A is separable, there is also a reasonable analogue of the Borel functions on T, namely the σ-closure $\mathcal{B}_A$ of A. In this paper we prove that $\mathcal{B}_A$ has an abundance of minimal projections, corresponding to points in T, and thus the notion of atomic and diffuse measures on T can be generalized to the non-commutative situation, since the diffuse measures are characterized as those measures that vanish at all points of T.

Let A be a separable C*-algebra and denote by $P$ the set of pure states of A. Choose in $P$ a maximal set $\{f_t : t \in T\}$ of pairwise inequivalent pure states of A. If $(\pi_t, H_t)$ denotes the irreducible representation of A corresponding to $f_t$, we define the reduced atomic representation $\rho$ of A as operators on the Hilbert space $H_a = \sum_t H_t$ by

$$\rho(x)(\sum_t \xi_t) = \sum_t \pi_t(x)\xi_t.$$ 

The reduced atomic representation is faithful and each pure state of A is a vector functional from $H_a$. Since any other choice of a maximal set in $P$ will give an equivalent representation, the reduced atomic representation is essentially unique. In particular the cardinality of T is uniquely determined as the cardinality of the set $\hat{A}$ of equivalence classes of irreducible representations of A. In what follows we shall identify A with its image $\rho(A)$.

Let $\mathcal{B}_A$ denote the monotone $\sigma$-closure of the self-adjoint part of A. Then $\mathcal{B}_A(= \mathcal{B}_a + i\mathcal{B}_a)$ is a C*-algebra in $B(H_a)$ called the Baire operators of A [14, Theorem 1]. Each representation $(\pi, H)$ of A extends to a $\sigma$-normal representation of $\mathcal{B}_A$ ([3, Theorem 3.2]) such that $\pi(\mathcal{B}_A)$ is the monotone $\sigma$-closure of $\pi(A)$ in $B(H)$ ([13, Proposition 4.2]). In particular, if $(\pi, H)$ is irreducible we have $\pi(\mathcal{B}_A) = B(H)$.

**Theorem 1.** There is a bijective correspondence between pure states of A and minimal projections of $\mathcal{B}_A$.
Proof. Since $A$ is separable, each point $f$ in $P$ is a closed $G_δ$
set. Hence there is a peaking element $x$ in $A^+$, with $\|x\| = 1$, such
that $f(x) = 1$ and $g(x) < 1$ for each state $g \neq f$ ([7, Theorem 9]). We
have $x^* \downarrow p$, where $p$ is a projection in $\mathcal{B}_A$. If $\xi$ is a unit vector
in $H_\alpha$ representing $f$, then $p\xi | \xi = 1$. For any unit vector $\eta$ in $H_\alpha$
which is not a multiple of $\xi$ we have $(\eta | \eta) \neq (\xi | \xi)$ for some $y$ in $A$;
hence $(p\eta | \eta) < 1$. It follows that $p$ is the one-dimensional projec-
ton on the subspace spanned by $\xi$, and consequently minimal.

If, conversely, $p$ is a minimal projection in $\mathcal{B}_A$, then $p\mathcal{B}_A p$ is a
commutative algebra, isomorphic with the complex field. The func-
tional $f$ on $\mathcal{B}_\alpha$, defined by $f(x) = pxp$, is the unique state extension
of the identity map on $p\mathcal{B}_A p$ ([11, Theorem 1.2]), which is pure; and
therefore $f$ is a $\sigma$-normal pure state of $\mathcal{B}_A$. But then $f \in P$. If $\xi$
is a unit vector in $pH_\alpha$, then $f(x) = (x\xi | \xi)$ and it follows from the
first part of the proof that $p$ is one-dimensional. Thus the corre-
spondence is bijective, and the theorem follows.

**Corollary 2.** There is a bijective correspondence between elements
in $\hat{A}$ and minimal projections in the center $C$ of $\mathcal{B}_A$.

**Remark.** Since $\mathcal{B}_A$ has a unit, we can identify $C$ with a $\sigma$-
closed algebra of bounded functions on $\hat{A}$. The projections in $C$ then
constitute the sets in a $\sigma$-field on $\hat{A}$, called the **Davies-Borel structure**
on $\hat{A}$. The above corollary tells us that points in $\hat{A}$ are Davies-Borel
sets (cf. [5, Theorem 2.9]).

Let $\mathcal{F}$ denote the smallest monotone closed $C^*$-subalgebra of
$\mathcal{B}_A$, which contains all minimal projections of $\mathcal{B}_A$. Then $\mathcal{F}$ can be
identified with the set of operators $x$ in the direct sum $\sum_{t \in \tau} B(H_t)$,
such that $x_t = 0$ except for countably many $t$ in $\tau$. In particular,
$\mathcal{F}$ is an ideal of $\mathcal{B}_A$.

**Definition.** A positive functional $f$ on $A$ is called **atomic** if
there is a projection $p$ in $\mathcal{F}$ such that $f(1 - p) = 0$; $f$ is called **diffuse**
if it vanishes at all minimal projections of $\mathcal{B}_A$.

**Proposition 3.** Each positive functional $f$ on $A$ has a unique
decomposition $f = f_a + f_d$ such that $f_a$ is atomic and $f_d$ is diffuse.
Moreover, $f_a$ and $f_d$ are centrally orthogonal.

**Proof.** Let $\alpha$ be the norm of the functional $f|\mathcal{F}$ on $\mathcal{F}$. There
is then a sequence $\{p_n\}$ in the unit ball of $\mathcal{F}^+$ such that $f(p_n) / \alpha$.
Replacing $p_n$ with its range projection, we may assume that all $p_n$
are projections. Let $p$ be the central support of $\vee p_n$. Then
and \( f(p) = \alpha \). Put \( f_a(x) = f(px) \) and \( f_d(x) = f((1 - p)x) \). Then \( f_a(1 - p) = 0 \), hence \( f_a \) is atomic; and for each \( x \) in \( \mathcal{F}^+ \), with \( \|x\| \leq 1 \), we have \( f(x(1 - p) + p) \leq \alpha \), hence \( f_d(x) = 0 \), and thus \( f_d \) is diffuse. By construction \( f_a \) and \( f_d \) are centrally orthogonal.

**Remark.** We see from the proof that a bounded functional will be atomic (respectively diffuse) if and only if the restriction to \( \mathcal{C} \) induces an atomic (respectively diffuse) measure on the Davies-Borel structure of \( \hat{A} \).

**Proposition 4.** A positive functional \( f \) on \( A \) is atomic exactly if it has the form \( f = \sum \alpha_n f_n \), with \( f_n \) in \( P \). Moreover, the summands can be chosen such that \( f_n \perp f_m \) for \( n \neq m \).

**Proof.** If \( f \) is atomic and \( f(1 - p) = 0 \) for a projection \( p \) in \( \mathcal{F} \), then, assuming that \( p \in \mathcal{C} \), we have \( p = \sum p_k \), where each \( p_k \) is a minimal projection in \( \mathcal{C} \). Thus \( f = \sum f_k \), where \( f_k(x) = f(p_k x) \), and each \( f_k \) is a \( \sigma \)-normal functional on \( B(H_k) = p_k \mathcal{B}_A \). There is then for each \( k \) an orthonormal basis \( \{\xi_{nk}\} \) for \( H_k \) and a sequence \( \{\alpha_{nk}\} \) of positive constants such that \( f_k(x) = \sum \alpha_{nk} \langle \xi_{nk}, x \rangle \xi_{nk} \), for all \( x \) in \( \mathcal{B}_A \). If \( f_{nk} \) denotes the pure state of \( A \) determined by \( \xi_{nk} \), then \( f = \sum \alpha_{nk} f_{nk} \), and since the \( f_{nk} \)'s are supported by pairwise orthogonal (minimal) projections in \( \mathcal{B}_A \), they are themselves orthogonal.

Conversely, if \( f = \sum \alpha_n f_n \), with all \( f_n \) in \( P \), then for each \( n \) let \( p_n \) be the minimal projection in \( \mathcal{B}_A \) such that \( f_n(p_n) = 1 \). Then \( p = \bigvee p_n \in \mathcal{F} \) and \( f(1 - p) = 0 \). Hence \( f \) is atomic, completing the proof.

**Definition.** An atom for a positive functional \( f \) on \( A \) is a projection \( p \) in \( \mathcal{B}_A \), such that \( f(p) > 0 \), but \( f(q)f(p - q) = 0 \), for any projection \( q \) in \( \mathcal{B}_A \) smaller than \( p \).

**Proposition 5.** A positive functional is diffuse exactly if it has no atoms.

**Proof.** Assume that \( p \) is an atom for \( f \). Then the state \( g \) of \( A \) given by \( g(x) = f(p)^{-1} f(px) \) is multiplicative, hence pure, on \( p \mathcal{B}_A p \). Since \( g \) is the unique state extension from \( p \mathcal{B}_A p \) to \( \mathcal{B}_A \) ([11, Theorem 1.2]), we conclude that \( g \in P \). There is then a minimal projection \( q \) in \( \mathcal{B}_A \) such that \( g(q) = 1 \). Since \( pq \mathcal{F} \), \( f \) is not diffuse.

Conversely, if \( f = f_a + f_d \), with \( f_a \neq 0 \), then from Proposition 4 there is a minimal projection \( p \) in \( \mathcal{B}_A \) such that \( f_a(p) > 0 \). Clearly \( p \) is an atom for \( f \), completing the proof.
The following proposition generalizes a well-known theorem from measure theory.

**PROPOSITION 6.** If $f$ is a diffuse functional on $A$ then, corresponding to each projection $p$ in $\mathcal{B}_A$ and each positive $\alpha < f(p)$, there is a projection $q$ in $\mathcal{B}_A$, with $q \leq p$ and $f(q) = \alpha$.

**Proof.** Since $p$ is not an atom for $f$, there is a projection $p_0 < p$ such that $0 < f(p_0) < f(p)$. Then either $f(p_0) \leq \frac{1}{2}f(p)$ or $f(p - p_0) \leq \frac{1}{2}f(p)$. Repeating this procedure we see that for any $\varepsilon > 0$ there is a projection $q_0 \leq p$ such that $0 < f(q_0) < \varepsilon$.

Now let $(\pi_f, H_f)$ be the representation of $A$ corresponding to $f$, and let $\xi_f$ be a vector in $H_f$ such that $f(x) = (\pi_f(x)\xi_f, \xi_f)$, for all $x$ in $\mathcal{B}_A$. Let $\{p_i\}$ be a maximal family of nonzero, orthogonal projections in $\pi_f(\mathcal{B}_A)$ such that $\Sigma p_i \leq \pi_f(p)$, and $\Sigma (p_i\xi_f, \xi_f) \leq \alpha$ for all finite sums. Since $\pi_f(\mathcal{B}_A)$ is a von Neumann algebra ([8, Theorem 2]), $p_0 = \Sigma p_i \in \pi_f(\mathcal{B}_A)$. It follows from spectral theory that there is a projection $q$ in $\mathcal{B}_A$, with $q \leq p$ and $\pi_f(q) = p_0$.

If we had $f(q) < \alpha$, then from the above we could find a projection $q_0 \leq p - q$, such that $0 < f(q_0) < \alpha - f(q)$. But then $\pi_f(q_0)$ could be adjoined to the family $\{p_i\}$, a contradiction. Therefore $f(q) = \alpha$, completing the proof.

For the sake of convenience we have stated all theorems in terms of bounded functionals. However, it is quite easy to extend the results to a large and important class of unbounded functionals.

Let $f$ be an extended valued, positive, σ-normal functional on $\mathcal{B}_A^+$ which is majorized by an invariant convex functional $\rho$ on $\mathcal{B}_A^+$ (see [13, §2] for definition). Assume furthermore that there is a sequence $\{e_n\}$ in $\mathcal{B}_A^+$ such that $\Sigma e_n = 1$, and $\rho(e_n) < \infty$ for each $n$. These conditions are satisfied if $f$ is a σ-finite σ-trace on $\mathcal{B}_A$ ([4], [5]) — take $\rho = f$ or $f$ is a $C^*$-integral of $A$ ([1, Proposition 4.4] and [13, Theorem 2.5]).

For each $x$ in $\mathcal{B}_A$ and each $n$, the element $e_n x$ belongs to the set of definition for $f$, and

$$|f(e_n x)|^2 \leq f(e_n)f(x^*e_n x) \leq \rho(e_n)\rho(x^*e_n x) \leq \|x\|^2\rho(e_n)^2.$$ 

If $f_n(x) = f(e_n x)$, then $\{f_n\}$ is a sequence of $\rho$-normal bounded functionals on $\mathcal{B}_A$ such that $f(x) = \Sigma f_n(x)$ for all $x$ in $\mathcal{B}_A^+$. For each $n$ there is a central projection $p_n$ in $\mathcal{F}$ such that $f_n(p_n \cdot)$ is atomic and $f_n(1 - p_n \cdot)$ is diffuse. Using $p = \vee p_n (\in \mathcal{C} \cap \mathcal{F})$ we see that Proposition 3 is valid for $f$.

To show that most of Proposition 4 holds also for an unbounded atomic functional $f$ of the above type, we notice that, as in the
proof of Proposition 4, we can write \( f = \sum \alpha f_k \), where each \( f_k \) is a \( \sigma \)-normal (unbounded) functional on an algebra \( B(H_k) \). If \( p \) is a one-dimensional projection in \( H_k \), then \( e_n^p \neq 0 \) for some \( n \) and therefore \( p = \| pe_n^p \|^{-1} pe_n^p \). It follows that
\[
\rho(p) = \| pe_n^p \|^{-1} \rho(e_n^2 pe_n^2) \leq \| pe_n^p \|^{-1} \rho(e_n) < \infty .
\]

This proves that \( f_k \) on \( B(H_k) \) is majorized by an invariant convex functional, which is finite at each operator in \( B(H_k) \) of finite rank. Thus \( f_k \) is a \( \mathcal{C}^* \)-integral on the \( \mathcal{C}^* \)-algebra of compact operators on \( H_k \), and from [10, Theorem 3.8] there is a \( b_k \) in \( B(H_k)^+ \) such that \( f_k(x) = \text{tr}(b_k x) \), for all \( x \) in \( B(H_k)^+ \). Hence \( f_k \) (and \( f \)) can be expressed as a sum \( \sum \alpha_n f_n \) with \( f_n \) in \( P \) for all \( n \) (and \( k \)). If each \( b_k \) can be diagonalized in \( B(H_k) \) then \( f \) can be written as a weighted sum of mutually orthogonal pure states. This is trivially the case if \( f \) is a trace, since then each \( f_k \) is a multiple of \( \text{tr} \). In general \( b_k \) can not be diagonalized, hence a decomposition in mutually orthogonal bounded functionals is not possible.

Proposition 5 and 6 can be generalized with the same ease. We leave the details to the reader.

Finally we notice that the condition of separability for the \( \mathcal{C}^* \)-algebra \( A \) is used primarily to ensure that \( \mathcal{B} \) is “large enough” under irreducible representations of \( A \). When \( A \) is nonseparable there may be irreducible representations of \( A \) on nonseparable Hilbert spaces. The pure states of \( A \) will then correspond to minimal projections in the Jordan algebra of Borel operators of \( A \) defined in [2, § 2.4]. This provides a method for studying atomic functionals on nonseparable \( \mathcal{C}^* \)-algebras. Another way is to define an atomic functional on \( A \) as one which is supported on a sequence of minimal projections from the enveloping von Neumann algebra of \( A \). But in this case the relation to measure theory becomes less clear.

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