

Pacific Journal of Mathematics

**ON THE OTHER SET OF THE BIORTHOGONAL
POLYNOMIALS SUGGESTED BY THE LAGUERRE
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Recently Konhauser considered the biorthogonal pair of polynomial sets $\{Z_n^\alpha(x; k)\}$ and $\{Y_n^\alpha(x; k)\}$ over $(0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$ and the basic polynomials x^k and x . For the polynomials $Y_n^\alpha(x; k)$, a generating function, some integral representations, two finite sum formulae, an infinite series and a generalized Rodrigues formula are obtained in this paper.

Biorthogonality and some other properties of $Z_n^\alpha(x; k)$ and $Y_n^\alpha(x; k)$ for any positive integer k were discussed by Konhauser ([1], [2]). For $k = 2$, the polynomials were discussed earlier by Preiser [4]. For $k = 1$, the polynomials $Y_n^\alpha(x; k)$, as also $Z_n^\alpha(x; k)$, reduce to the generalized Laguerre polynomials $L_n^\alpha(x)$.

In a recent paper [3], we obtained generating functions and other results for the polynomials $Z_n^\alpha(x; k)$ in x^k . The present paper is concerned only with the polynomials $Y_n^\alpha(x; k)$ in x which form the other set of the biorthogonal pair. The results of the paper reduce, when $k = 1$, to some standard properties of $L_n^\alpha(x)$. Simplicity of the procedure for deriving the generating relation (2.1) which may be regarded as our principal result, seems to be of some passing interest.

2. A generating function for $Y_n^\alpha(x; k)$. We begin with the contour integral representation [2, (26)]

$$(2.1) \quad Y_n^\alpha(x; k) = (k/2\pi i) \int_C e^{-xt} (t+1)^{\alpha+kn} [(t+1)^k - 1]^{-(n+1)} dt$$

where we take C as a closed contour enclosing $t = 0$ and lying within $|t| < 1$. If we make the substitution $u = 1 - (t+1)^{-k}$, we get another integral representation for $Y_n^\alpha(x; k)$, viz.

$$(2.2) \quad Y_n^\alpha(x; k) = (2\pi i)^{-1} \int_{C'} (1-u)^{-(\alpha+1)/k} \exp[x\{1 - (1-u)^{-1/k}\}] u^{-n-1} du$$

C' being a circle with centre $u = 0$ and a small radius. By standard arguments of complex analysis we obtain the generating relation

$$(2.3) \quad \sum_{n=0}^{\infty} Y_n^\alpha(x; k) u^n = (1-u)^{-(\alpha+1)/k} \exp[x\{1 - (1-u)^{-1/k}\}]$$

for $\text{Re}(\alpha + 1) > 0$, $|u| < 1$ and positive integers k .

Since the generating relation (2.3) is of the form

$$A(u) \exp [xH(u)] = \sum_{n=0}^{\infty} Y_n^\alpha(x; k) u^n,$$

it at once follows ([6], [5]) that the set $\{Y_n^\alpha(x; k)\}$ is of Sheffer A -type zero. One of the several immediate consequences of this fact [5, Theorems 73-76] is that there exists a sequence $\{h_i\}$ independent of x and n such that

$$(2.4) \quad DY_n^\alpha(x; k) = \sum_{m=0}^{n-1} h_m Y_{n-m-1}^\alpha(x; k).$$

In (2.2) putting $s = x^k(1-u)^{-1}$, we are led to still another integral representation

$$(2.5) \quad Y_n^\alpha(x; k) = (2\pi i)^{-1} e^x x^{k-\alpha-1} \int_{\sigma} s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})(s-x^k)^{-n-1} ds$$

where σ denotes the circle $|s-x^k|=r$ with small r . Evidently σ may be any small closed contour encircling $s=x^k$.

Evaluating the integral in (2.5) by the residue theorem, we obtain a generalized Rodrigues formula:

$$(2.6) \quad Y_n^\alpha(x; k) = (n!)^{-1} e^x x^{k-\alpha-1} [D^n s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})]_{s=x^k}.$$

For $k=1$, it reduces to the Rodrigues formula for $L_n^\alpha(x)$.

3. Applications. In this section we apply the generating relation of the previous section to obtain two finite sum formulae for $Y_n^\alpha(x; k)$ and also to prove a result involving an infinite series of these polynomials.

a. Two finite sums involving $Y_n^\alpha(x; k)$. From the generating relation (2.3) and the simple relation

$$(1-u)^{-(\alpha+1)/k} = (1-u)^{-(\beta+1)/k} \sum_{m=0}^{\infty} (m!)^{-1} \left(\frac{\alpha-\beta}{k}\right)_m u^m,$$

it follows that

$$(3.1) \quad Y_n^\alpha(x; k) = \sum_{m=0}^n (m!)^{-1} \left(\frac{\alpha-\beta}{k}\right)_m Y_{n-m}^\beta(x; k)$$

where α and β are arbitrary.

Also from (2.3), on using

$$\begin{aligned} & (1-u)^{-((\alpha+\beta+1)+1)/k} \exp[(x+y)\{1-(1-u)^{-1/k}\}] \\ &= (1-u)^{-(\alpha+1)/k} \exp[x\{1-(1-u)^{-1/k}\}] \cdot (1-u)^{-(\beta+1)/k} \\ & \quad \times \exp[y\{1-(1-u)^{-1/k}\}] \end{aligned}$$

we get that

$$(3.2) \quad Y_n^{\alpha+\beta+1}(x+y; k) = \sum_{m=0}^n Y_m^\alpha(x; k) Y_{n-m}^\beta(y; k)$$

for arbitrary α and β .

b. A series of polynomials $Y_n^\alpha(x; k)$. We show that

$$(3.3) \quad \sum_{n=0}^\infty \frac{(n+m)!}{n! m!} Y_{n+m}^\alpha(x; k) u^n = (1-u)^{-(\alpha+m k+1)/k} \exp [x\{1-(1-u)^{-1/k}\}] Y_m^\alpha(x(1-u)^{-1/k}; k).$$

Using the obvious result

$$1-u-v = (1-u)\{1-v(1-u)^{-1}\}$$

we have that

$$\begin{aligned} F(u, v) &\equiv (1-u-v)^{-(\alpha+1)/k} \exp [x\{1-(1-u-v)^{-1/k}\}] \\ &= (1-u)^{-(\alpha+1)/k} \exp [x\{1-(1-u)^{-1/k}\}] \cdot (1-v(1-u)^{-1})^{-(\alpha+1)/k} \\ &\quad \cdot \exp [x(1-u)^{-1/k}\{1-(1-v(1-u)^{-1})^{-1/k}\}] \\ &= (1-u)^{-(\alpha+1)/k} \exp [x\{1-(1-u)^{-1/k}\}] \\ &\quad \cdot \sum_{m=0}^\infty Y_m^\alpha(x(1-u)^{-1/k}; k) [v(1-u)^{-1}]^m, \end{aligned}$$

applying (2.3). But using (2.3), we also find that

$$\begin{aligned} F(u, v) &= \sum_{n=0}^\infty Y_n^\alpha(x; k) (u+v)^n \\ &= \sum_{n=0}^\infty \sum_{m=0}^n \frac{n!}{m!(n-m)!} u^{n-m} v^m Y_n^\alpha(x; k) \\ &= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(m+n)!}{m! n!} Y_{n+m}^\alpha(x; k) u^n v^m. \end{aligned}$$

Comparing the coefficients of v^m in the two expansions obtained for $F(u, v)$, we obtain (3.3).

This result is analogous to a property possessed by almost all the classical orthogonal polynomials [5; 95(7), 111(1), 120(9), 144(23)] except possibly by the Jacobi polynomials.

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