

# Pacific Journal of Mathematics

**ON THE OTHER SET OF THE BIORTHOGONAL  
POLYNOMIALS SUGGESTED BY THE LAGUERRE  
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# ON THE OTHER SET OF THE BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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**Recently Konhauser considered the biorthogonal pair of polynomial sets  $\{Z_n^\alpha(x; k)\}$  and  $\{Y_n^\alpha(x; k)\}$  over  $(0, \infty)$  with respect to the weight function  $x^\alpha e^{-x}$  and the basic polynomials  $x^k$  and  $x$ . For the polynomials  $Y_n^\alpha(x; k)$ , a generating function, some integral representations, two finite sum formulae, an infinite series and a generalized Rodrigues formula are obtained in this paper.**

Biorthogonality and some other properties of  $Z_n^\alpha(x; k)$  and  $Y_n^\alpha(x; k)$  for any positive integer  $k$  were discussed by Konhauser ([1], [2]). For  $k = 2$ , the polynomials were discussed earlier by Preiser [4]. For  $k = 1$ , the polynomials  $Y_n^\alpha(x; k)$ , as also  $Z_n^\alpha(x; k)$ , reduce to the generalized Laguerre polynomials  $L_n^\alpha(x)$ .

In a recent paper [3], we obtained generating functions and other results for the polynomials  $Z_n^\alpha(x; k)$  in  $x^k$ . The present paper is concerned only with the polynomials  $Y_n^\alpha(x; k)$  in  $x$  which form the other set of the biorthogonal pair. The results of the paper reduce, when  $k = 1$ , to some standard properties of  $L_n^\alpha(x)$ . Simplicity of the procedure for deriving the generating relation (2.1) which may be regarded as our principal result, seems to be of some passing interest.

**2. A generating function for  $Y_n^\alpha(x; k)$ .** We begin with the contour integral representation [2, (26)]

$$(2.1) \quad Y_n^\alpha(x; k) = (k/2\pi i) \int_C e^{-xt}(t+1)^{\alpha+kn} [(t+1)^k - 1]^{-(n+1)} dt$$

where we take  $C$  as a closed contour enclosing  $t = 0$  and lying within  $|t| < 1$ . If we make the substitution  $u = 1 - (t+1)^{-k}$ , we get another integral representation for  $Y_n^\alpha(x; k)$ , viz.

$$(2.2) \quad Y_n^\alpha(x; k) = (2\pi i)^{-1} \int_{C'} (1-u)^{-(\alpha+1)/k} \exp [x\{1 - (1-u)^{-1/k}\}] u^{n-1} du$$

$C'$  being a circle with centre  $u = 0$  and a small radius. By standard arguments of complex analysis we obtain the generating relation

$$(2.3) \quad \sum_{n=0}^{\infty} Y_n^\alpha(x; k) u^n = (1-u)^{-(\alpha+1)/k} \exp [x\{1 - (1-u)^{-1/k}\}]$$

for  $\text{Re}(\alpha + 1) > 0$ ,  $|u| < 1$  and positive integers  $k$ .

Since the generating relation (2.3) is of the form

$$A(u) \exp [xH(u)] = \sum_{n=0}^{\infty} Y_n^\alpha(x; k) u^n,$$

it at once follows ([6], [5]) that the set  $\{Y_n^\alpha(x; k)\}$  is of Sheffer  $A$ -type zero. One of the several immediate consequences of this fact [5, Theorems 73–76] is that there exists a sequence  $\{h_i\}$  independent of  $x$  and  $n$  such that

$$(2.4) \quad DY_n^\alpha(x; k) = \sum_{m=0}^{n-1} h_m Y_{n-m-1}^\alpha(x; k).$$

In (2.2) putting  $s = x^k(1 - u)^{-1}$ , we are led to still another integral representation

$$(2.5) \quad Y_n^\alpha(x; k) = (2\pi i)^{-1} e^x x^{k-\alpha-1} \int_{\sigma} s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})(s - x^k)^{-n-1} ds$$

where  $\sigma$  denotes the circle  $|s - x^k| = r$  with small  $r$ . Evidently  $\sigma$  may be any small closed contour encircling  $s = x^k$ .

Evaluating the integral in (2.5) by the residue theorem, we obtain a generalized Rodrigues formula:

$$(2.6) \quad Y_n^\alpha(x; k) = (n!)^{-1} e^x x^{k-\alpha-1} [D^n s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})]_{s=x^k}.$$

For  $k = 1$ , it reduces to the Rodrigues formula for  $L_n^\alpha(x)$ .

**3. Applications.** In this section we apply the generating relation of the previous section to obtain two finite sum formulae for  $Y_n^\alpha(x; k)$  and also to prove a result involving an infinite series of these polynomials.

**a. Two finite sums involving  $Y_n^\alpha(x; k)$ .** From the generating relation (2.3) and the simple relation

$$(1 - u)^{-(\alpha+1)/k} = (1 - u)^{-(\beta+1)/k} \sum_{m=0}^{\infty} (m!)^{-1} \left(\frac{\alpha - \beta}{k}\right)_m u^m,$$

it follows that

$$(3.1) \quad Y_n^\alpha(x; k) = \sum_{m=0}^n (m!)^{-1} \left(\frac{\alpha - \beta}{k}\right)_m Y_{n-m}^\beta(x; k)$$

where  $\alpha$  and  $\beta$  are arbitrary.

Also from (2.3), on using

$$\begin{aligned} & (1 - u)^{-\{(\alpha+\beta+1)+1\}/k} \exp [(x + y)\{1 - (1 - u)^{-1/k}\}] \\ &= (1 - u)^{-(\alpha+1)/k} \exp [x\{1 - (1 - u)^{-1/k}\}] \cdot (1 - u)^{-(\beta+1)/k} \\ & \quad \times \exp [y\{1 - (1 - u)^{-1/k}\}] \end{aligned}$$

we get that

$$(3.2) \quad Y_n^{\alpha+\beta+1}(x+y; k) = \sum_{m=0}^n Y_m^\alpha(x; k) Y_{n-m}^\beta(y; k)$$

for arbitrary  $\alpha$  and  $\beta$ .

b. A series of polynomials  $Y_n^\alpha(x; k)$ . We show that

$$(3.3) \quad \sum_{n=0}^{\infty} \frac{(n+m)!}{n! m!} Y_{n+m}^\alpha(x; k) u^n = (1-u)^{-(\alpha+m k+1)/k} \exp[x\{1-(1-u)^{-1/k}\}] Y_m^\alpha(x(1-u)^{-1/k}; k).$$

Using the obvious result

$$1-u-v = (1-u)\{1-v(1-u)^{-1}\}$$

we have that

$$\begin{aligned} F(u, v) &\equiv (1-u-v)^{-(\alpha+1)/k} \exp[x\{1-(1-u-v)^{-1/k}\}] \\ &= (1-u)^{-(\alpha+1)/k} \exp[x\{1-(1-u)^{-1/k}\}] \cdot (1-v(1-u)^{-1})^{-(\alpha+1)/k} \\ &\quad \cdot \exp[x(1-u)^{-1/k}\{1-(1-v(1-u)^{-1})^{-1/k}\}] \\ &= (1-u)^{-(\alpha+1)/k} \exp[x\{1-(1-u)^{-1/k}\}] \\ &\quad \cdot \sum_{m=0}^{\infty} Y_m^\alpha(x(1-u)^{-1/k}; k) [v(1-u)^{-1}]^m, \end{aligned}$$

applying (2.3). But using (2.3), we also find that

$$\begin{aligned} F(u, v) &= \sum_{n=0}^{\infty} Y_n^\alpha(x; k) (u+v)^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{n!}{m!(n-m)!} u^{n-m} v^m Y_n^\alpha(x; k) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n)!}{m! n!} Y_{n+m}^\alpha(x; k) u^n v^m. \end{aligned}$$

Comparing the coefficients of  $v^m$  in the two expansions obtained for  $F(u, v)$ , we obtain (3.3).

This result is analogous to a property possessed by almost all the classical orthogonal polynomials [5; 95(7), 111(1), 120(9), 144(23)] except possibly by the Jacobi polynomials.

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