ON THE OTHER SET OF THE BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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Recently Konhauser considered the biorthogonal pair of polynomial sets \{Z_l(x; k)\} and \{Y_l(x; k)\} over \((0, \infty)\) with respect to the weight function \(x^a e^{-x}\) and the basic polynomials \(x^k\) and \(x\). For the polynomials \(Y_l^a(x; k)\), a generating function, some integral representations, two finite sum formulae, an infinite series and a generalized Rodrigues formula are obtained in this paper.

Biorthogonality and some other properties of \(Z_l^a(x; k)\) and \(Y_l^a(x; k)\) for any positive integer \(k\) were discussed by Konhauser ([1], [2]). For \(k = 2\), the polynomials were discussed earlier by Preiser [4]. For \(k = 1\), the polynomials \(Y_l^a(x; k)\), as also \(Z_l^a(x; k)\), reduce to the generalized Laguerre polynomials \(L_l^a(x)\).

In a recent paper [3], we obtained generating functions and other results for the polynomials \(Z_l^a(x; k)\) in \(x^k\). The present paper is concerned only with the polynomials \(Y_l^a(x; k)\) in \(x\) which form the other set of the biorthogonal pair. The results of the paper reduce, when \(k = 1\), to some standard properties of \(L_l^a(x)\). Simplicity of the procedure for deriving the generating relation (2.1) which may be regarded as our principal result, seems to be of some passing interest.

2. A generating function for \(Y_l^a(x; k)\). We begin with the contour integral representation [2, (26)]

\[
(2.1) \quad Y_l^a(x; k) = \left(\frac{k}{2\pi i}\right) \int_C e^{-xt}(t + 1)^{\alpha+k}[t + 1 - 1]^{-(\alpha+1)} dt
\]

where we take \(C\) as a closed contour enclosing \(t = 0\) and lying within \(|t| < 1\). If we make the substitution \(u = 1 - (t+1)^{-k}\), we get another integral representation for \(Y_l^a(x; k)\), viz.

\[
(2.2) \quad Y_l^a(x; k) = (2\pi i)^{-1} \int_{C'} (1 - u)^{-(\alpha+1)/k} \exp[x(1 - (1 - u)^{-1/k})] u^{\alpha-1} du
\]

\(C'\) being a circle with centre \(u = 0\) and a small radius. By standard arguments of complex analysis we obtain the generating relation

\[
(2.3) \quad \sum_{n=0}^{\infty} Y_n^a(x; k) u^n = (1 - u)^{-(\alpha+1)/k} \exp[x(1 - (1 - u)^{-1/k})]
\]

for \(\text{Re} (\alpha + 1) > 0\), \(|u| < 1\) and positive integers \(k\).
Since the generating relation (2.3) is of the form

\[ A(u) \exp [xH(u)] = \sum_{n=0}^{\infty} Y_n^\alpha(x; k)u^n, \]

it at once follows ([6], [5]) that the set \( \{Y_n^\alpha(x; k)\} \) is of Sheffer A-type zero. One of the several immediate consequences of this fact [5, Theorems 73-76] is that there exists a sequence \( \{h_i\} \) independent of \( x \) and \( n \) such that

\[ \frac{D}{dx} Y_n^\alpha(x; k) = \sum_{m=0}^{n-1} h_m Y_{n-m}^\alpha(x; k). \]

In (2.2) putting \( s = x^k(1 - u)^{-1} \), we are led to still another integral representation

\[ Y_n^\alpha(x; k) = (2\pi i)^{-1}e^{x^k(1 - u)^{-1}}\int_{\sigma} e^{s^{n-1}}s^n e^{-(s^{1/k})(s - x^k)^{-n-1}} ds \]

where \( \sigma \) denotes the circle \( |s - x^k| = r \) with small \( r \). Evidently \( \sigma \) may be any small closed contour encircling \( s = x^k \).

Evaluating the integral in (2.5) by the residue theorem, we obtain a generalized Rodrigues formula:

\[ Y_n^\alpha(x; k) = (n!)^{-1}e^{x^k(1 - u)^{-1}} [D^n e^{-(s^{1/k})}]_{s=x^k}. \]

For \( k = 1 \), it reduces to the Rodrigues formula for \( L_n^\alpha(x) \).

3. Applications. In this section we apply the generating relation of the previous section to obtain two finite sum formulae for \( Y_n^\alpha(x; k) \) and also to prove a result involving an infinite series of these polynomials.

a. Two finite sums involving \( Y_n^\alpha(x; k) \). From the generating relation (2.3) and the simple relation

\[ (1 - u)^{-(\alpha + 1)/k} = (1 - u)^{-(\beta + 1)/k} \sum_{m=0}^{\infty} (m!)^{-1}\left(\frac{\alpha - \beta}{k}\right)_m u^m, \]

if follows that

\[ Y_n^\alpha(x; k) = \sum_{m=0}^{n} (m!)^{-1}\left(\frac{\alpha - \beta}{k}\right)_m Y_{n-m}^\beta(x; k) \]

where \( \alpha \) and \( \beta \) are arbitrary.

Also from (2.3), on using

\[
(1 - u)^{-(\alpha + \beta + 1)/k} \exp [(x + y)(1 - (1 - u)^{-1/k})] \\
= (1 - u)^{-(\alpha + 1)/k} \exp [x(1 - (1 - u)^{-1/k})] \cdot (1 - u)^{-(\beta + 1)/k} \\
\times \exp [y(1 - (1 - u)^{-1/k})]
\]

we get that
for arbitrary $\alpha$ and $\beta$.

b. A series of polynomials $Y^\alpha_n(x; k)$. We show that

$$
\sum_{m=0}^{\infty} \frac{(n + m)!}{n! m!} Y^\alpha_{n+m}(x; k) u^n
$$

$$
= (1 - u)^{-\alpha+1/k} \exp [x(1 - (1 - u)^{-1/k})] Y^\alpha_n(x(1 - u)^{-1/k}; k).
$$

Using the obvious result

$$
1 - u - v = (1 - u)(1 - v(1 - u)^{-1})
$$

we have that

$$
F(u, v) \equiv (1 - u - v)^{-\alpha+1/k} \exp [x(1 - (1 - u - v)^{-1/k})]
$$

$$
= (1 - u)^{-\alpha+1/k} \exp [x(1 - (1 - u)^{-1/k})] \cdot (1 - v(1 - u)^{-1})^{-\alpha+1/k} \exp [x(1 - u)^{-1/k}(1 - (1 - v(1 - u)^{-1})^{-1/k})]$$

$$
= (1 - u)^{-\alpha+1/k} \exp [x(1 - (1 - u)^{-1/k})] \cdot \sum_{m=0}^{\infty} Y^\alpha_m(x(1 - u)^{-1/k}; k)[v(1 - u)^{-1}]^m,
$$

applying (2.3). But using (2.3), we also find that

$$
F(u, v) = \sum_{n=0}^{\infty} Y^\alpha_n(x; k)(u + v)^n
$$

$$
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{n!}{m!(n - m)!} u^{n-m} v^m Y^\alpha_n(x; k)
$$

$$
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m + n)!}{m! n!} Y^\alpha_{n+m}(x; k) u^m v^m.
$$

Comparing the coefficients of $v^m$ in the two expansions obtained for $F(u, v)$, we obtain (3.3).

This result is analogous to a property possessed by almost all the classical orthogonal polynomials [5; 95(7), 111(1), 120(9), 144(23)] except possibly by the Jacobi polynomials.

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Mohammad Shafqat Ali and Marvin David Marcus, *On the degree of the minimal polynomial of a commutator operator* ........................................... 561
Howard Anton and William J. Pervin, *Integration on topological semifields* ................................................................. 567
Martin Bartelt, *Multipliers and operator algebras on bounded analytic functions* ......................................................... 575
Donald Earl Bennett, *Aposyndetic properties of unicoherent continua* ............... 585
James W. Bond, *Lie algebras of genus one and genus two* ........................................ 591
Mario Borelli, *The cohomology of divisorial varieties* .................................................. 617
Carlos R. Borges, *How to recognize homeomorphisms and isometries* ............ 625
J. C. Breckenridge, *Burkill-Cesari integrals of quasi additive interval functions* ......................................................... 635
J. Csima, *A class of counterexamples on permanents* .................................................. 655
Carl Hanson Fitzgerald, *Conformal mappings onto ω-swirly domains* ............. 657
Newcomb Greenleaf, *Analytic sheaves on Klein surfaces* ........................................ 671
G. Goss and Giovanni Viglino, *C-compact and functionally compact spaces* ................................................................. 677
Charles Lemuel Hagopian, *Arcwise connectivity of semi-aposyndetic plane continua* ........................................................................................................ 683
John Harris and Olga Higgins, *Prime generators with parabolic limits* ............... 687
David Michael Henry, *Stratifiable spaces, semi-stratifiable spaces, and their relation through mappings* ......................................................... 697
Raymond D. Holmes, *On contractive semigroups of mappings* ......................... 701
Shûichirô Maeda, *On point-free parallelism and Wilcox lattices* ....................... 725
Gary L. Musser, *Linear semiprime (p; q) radicals* ........................................ 749
William Charles Nemitz and Thomas Paul Whaley, *Varieties of implicative semilattices* ........................................................................................................ 759
Jaroslav Nešetřil, *A congruence theorem for asymmetric trees* ......................... 771
Robert Anthony Nowlan, *A study of H-spaces via left translations* ................. 779
Gert Kjærgaard Pedersen, *Atomic and diffuse functionals on a C*-algebra* ...... 795
Tilak Raj Prabhakar, *On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials* ......................................................... 801
Leland Edward Rogers, *Mutually aposyndetic products of chainable continua* ........................................................................................................ 805
Frederick Stern, *An estimate for Wiener integrals connected with squared error in a Fourier series approximation* ......................................................... 813
Leonard Paul Sternbach, *On k-shrinking and k-boundedly complete basic sequences and quasi-reflexive spaces* ......................................................... 817
Pak-Ken Wong, *Modular annihilator A*-algebras ........................................ 825