AN ESTIMATE FOR WIENER INTEGRALS CONNECTED WITH
SQUARED ERROR IN A FOURIER SERIES APPROXIMATION

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If a function $x(\sigma)$, $0 \leq \sigma \leq t$, is in Lip-$\alpha$, $0 < \alpha < 1$, $x(0) = 0$ and if $c_k$ $(k = 0, 1, 2, \cdots)$ are its Fourier coefficients with respect to the functions $\sqrt{2/t} \sin [\pi(k + \frac{1}{2})\sigma/t]$, then it is known [1, pp. 171–172] that

$$
\sum_{k \geq n} c_k^2 \leq \frac{A}{(n + \frac{1}{2})^{2\alpha}}, \quad n \geq 0
$$

where $A$ is a positive number not depending on $n$. We will show a connection between this estimate and an estimate for Wiener integrals. Let $E_w \{ \}$ denote expectation on a Wiener process, that is, a Gaussian process with mean function zero, covariance function $\min(\sigma, \tau)$, $0 \leq \sigma, \tau \leq t$ and sample functions $z(\sigma)$ with $z(0) = 0$.

THEOREM: Let $x(\sigma)$ be in $C[0, t]$ and let $c_k$ be the Fourier coefficients of $x(\sigma)$ with respect to the normalized eigenfunctions associated with $\min(\sigma, \tau)$. That is

$$
c_k = \sqrt{\frac{2}{t}} \int_0^t x(\sigma) \sin [\pi(k + \frac{1}{2})\sigma/t]d\sigma.
$$

Let $0 < \alpha < 1$. Then estimate (1) is a necessary and sufficient condition for the estimate

$$
e^{-v/2} \int_0^t x^2(\sigma) d\sigma \leq \frac{E_w \left\{ e^{-\alpha^2/2} \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\}}{E_w \left\{ e^{-v/2} \int_0^t z^2(\sigma) d\sigma \right\}}
$$

for all positive $v$, where $B$ is a positive number not depending on $v$.

Proof. From Cameron and Donsker’s proof of a lemma [2, p. 27–28], we have that, for the case $\rho_k = [\pi(k + \frac{1}{2})/t]^\alpha$, the right side of (2) equals

$$
e^{-\alpha^2/2} \sum_{k=0}^\infty \frac{c_k^2 \rho_k}{\rho_k + v}.
$$

Hence estimate (2) holds if and only if

$$
\sum_{k=0}^\infty \frac{c_k^2 \rho_k}{\rho_k + v} \leq \frac{B}{v^\alpha}
$$
for all positive \( v \). To prove that (2) implies (1) note that for each fixed value of \( v \), as \( k \to \infty \), \( [\rho_k | (\rho_k + v)] \uparrow 1 \). Therefore for each \( n \), by the remark and (3),

\[
\frac{\rho_n}{\rho_n + v} \sum_{k \geq n} c_k^2 \leq \sum_{k \geq n} \frac{c_k^2 \rho_k}{\rho_k + v} \leq \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + v} \leq \frac{B}{v^n}
\]

for all positive \( v \). Letting \( v = \rho_n \) we have

\[
\sum_{k \geq n} c_k^2 \leq \frac{2B}{[\pi(n + \frac{1}{2})/t]^{2\alpha}} = \frac{A}{(n + \frac{1}{2})^{2\alpha}}
\]

which is estimate (1).

We now show that the latter estimate implies (3). Since the left side of (3) is bounded by \( \sum_{k=0}^{\infty} c_k^2 \), estimate (3) holds for \( 0 < v \leq 1 \). Hence it suffices to prove (3) for \( v > 1 \). To simplify notation set

\[
S(n) = \sum_{k \geq n} c_k^2 \leq \frac{A}{(n + \frac{1}{2})^{2\alpha}}
\]

by hypothesis. For any \( n \geq 1 \)

\[
\sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{(\rho_k + v)} = \sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + v)} + S(n)\frac{\rho_n}{\rho_n + v} + \sum_{k=n+1}^{\infty} S(k) \left[ \frac{\rho_k}{\rho_k + v} - \frac{\rho_{k-1}}{\rho_{k-1} + v} \right].
\]

For the first two terms on the right side of (5) we have

\[
\sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + v)} + S(n)\frac{\rho_n}{\rho_n + v} \leq \frac{2S(0)\rho_n}{\rho_n + v} < \frac{2S(0)\rho_n}{v^n}.
\]

To estimate the third term consider first

\[
\left[ \frac{\rho_k}{\rho_k + v} - \frac{\rho_{k-1}}{\rho_{k-1} + v} \right] = \frac{\nu(\rho_k - \rho_{k-1})}{(\rho_k + v)(\rho_{k-1} + v)}.
\]

Since \( \rho_k - \rho_{k-1} = 2(\pi/t)^2 k \) and \( (\rho_k + v)(\rho_{k-1} + v) \geq [(\pi k/2t)^2 + v]^2 \), the right side of (7) is dominated by \( 2(\pi/t)^2 \nu k[(\pi k/2t)^2 + v]^{-2} \). Applying (4) and the above, we have

\[
\sum_{k=n+1}^{\infty} S(k) \left[ \frac{\rho_k}{\rho_k + v} - \frac{\rho_{k-1}}{\rho_{k-1} + v} \right] \leq 2(\pi/t)^2 \nu \sum_{k=n+1}^{\infty} \frac{k^{2\alpha-2\alpha}}{[(\pi k/2t)^2 + v]^2}.
\]

To get the desired estimate we will use standard integral estimates. For \( \alpha \geq 1/2 \), the summands in the right side of (8) decrease monotonically with \( k \) for fixed \( v \). If \( \alpha < 1/2 \), the function

\[
g(\tilde{\xi}) = \tilde{\xi}^{1-2\alpha} \left[ \left( \frac{\pi^2}{2t} \right)^2 + v \right]^{-2}.
\]

\( \tilde{\xi} \geq 0 \)
has a unique local and absolute maximum at
\[ \xi^* = \frac{2t}{\pi} \left( \frac{1 - 2\alpha}{3 + 2\alpha} \right)^{1/2}. \]

In this case if \( n \geq \xi^* \), the summands in the right side of (8) decrease monotonically as \( k \) increases and
\[
2 \left( \frac{\pi}{t} \right)^2 A \nu \sum_{k=n+1}^{\infty} \frac{k^{1-2\alpha}}{[(\pi k/2t)^2 + \nu]^2} \leq \frac{2(\pi/t)^2 A}{\nu} \int_{\xi^*}^{\infty} \frac{\xi^{1-2\alpha}}{[(\pi \xi/2t \sqrt{\nu})^2 + 1]^2} d\xi \\
= \frac{8(\pi/2t)^2 A}{\nu^2} \int_{\xi^*/2t \sqrt{\nu}}^{\infty} \frac{\eta^{1-2\alpha}}{(\eta^2 + 1)^2} d\eta \\
< 8 \left( \frac{\pi}{2t} \right)^2 A \int_{0}^{\infty} \frac{\eta^{1-2\alpha}}{(\eta^2 + 1)^2} d\eta \frac{1}{\nu^2}.
\]

In the case \( \alpha \geq \frac{1}{2} \), (9) holds for any \( n \) and in both cases the last integral converges since \( 0 < \alpha < 1 \). To complete the proof we fix in (5) \( n = n^* \geq \xi^* \) in the case \( \alpha \leq \frac{1}{2} \) or \( n = n^* \geq 1 \) if \( \alpha \geq \frac{1}{2} \). Estimates (6), (8), and (9) complete the proof.

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