

# Pacific Journal of Mathematics

**AN ESTIMATE FOR WIENER INTEGRALS CONNECTED WITH  
SQUARED ERROR IN A FOURIER SERIES APPROXIMATION**

FREDERICK STERN

## AN ESTIMATE FOR WIENER INTEGRALS CONNECTED WITH SQUARED ERROR IN A FOURIER SERIES APPROXIMATION

FREDERICK STERN

If a function  $x(\sigma)$ ,  $0 \leq \sigma \leq t$ , is in Lip- $\alpha$ ,  $0 < \alpha < 1$ ,  $x(0) = 0$  and if  $c_k$  ( $k = 0, 1, 2, \dots$ ) are its Fourier coefficients with respect to the functions  $\sqrt{2/t} \sin [\pi(k + \frac{1}{2})\sigma/t]$ , then it is known [1, pp. 171-172] that

$$(1) \quad \sum_{k \geq n} c_k^2 \leq \frac{A}{(n + \frac{1}{2})^{2\alpha}}, \quad n \geq 0$$

where  $A$  is a positive number not depending on  $n$ . We will show a connection between this estimate and an estimate for Wiener integrals. Let  $E_w\{ \}$  denote expectation on a Wiener process, that is, a Gaussian process with mean function zero, covariance function  $\min(\sigma, \tau)$ ,  $0 \leq \sigma, \tau \leq t$  and sample functions  $z(\sigma)$  with  $z(0) = 0$ .

**THEOREM:** Let  $x(\sigma)$  be in  $C[0, t]$  and let  $c_k$  be the Fourier coefficients of  $x(\sigma)$  with respect to the normalized eigenfunctions associated with  $\min(\sigma, \tau)$ . That is

$$c_k = \sqrt{\frac{2}{t}} \int_0^t x(\sigma) \sin [\pi(k + \frac{1}{2})\sigma/t] d\sigma.$$

Let  $0 < \alpha < 1$ . Then estimate (1) is a necessary and sufficient condition for the estimate

$$(2) \quad e^{-(B/2)\nu^{1-\alpha}} \leq \frac{E_W \left\{ e^{-(\nu/2) \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma} \right\}}{E_W \left\{ e^{-(\nu/2) \int_0^t z^2(\sigma) d\sigma} \right\}}$$

for all positive  $\nu$ , where  $B$  is a positive number not depending on  $\nu$ .

*Proof.* From Cameron and Donsker's proof of a lemma [2, p. 27-28], we have that, for the case  $\rho_k = [\pi(k + \frac{1}{2})/t]^2$ , the right side of (2) equals

$$e^{-\nu/2} \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu}.$$

Hence estimate (2) holds if and only if

$$(3) \quad \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leq \frac{B}{\nu^\alpha}$$

for all positive  $\nu$ . To prove that (2) implies (1) note that for each fixed value of  $\nu$ , as  $k \rightarrow \infty$ ,  $[\rho_k | (\rho_k + \nu)] \uparrow 1$ . Therefore for each  $n$ , by the remark and (3),

$$\frac{\rho_n}{\rho_n + \nu} \sum_{k \geq n} c_k^2 \leq \sum_{k \geq n} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leq \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leq \frac{B}{\nu^\alpha}$$

for all positive  $\nu$ . Letting  $\nu = \rho_n$  we have

$$\sum_{k \geq n} c_k^2 \leq \frac{2B}{[\pi(n + \frac{1}{2})/t]^{2\alpha}} = \frac{A}{(n + \frac{1}{2})^{2\alpha}}$$

which is estimate (1).

We now show that the latter estimate implies (3). Since the left side of (3) is bounded by  $\sum_{k=0}^{\infty} c_k^2$ , estimate (3) holds for  $0 < \nu \leq 1$ . Hence it suffices to prove (3) for  $\nu > 1$ . To simplify notation set

$$(4) \quad S(n) = \sum_{k \geq n} c_k^2 \leq \frac{A}{(n + \frac{1}{2})^{2\alpha}}$$

by hypothesis. For any  $n \geq 1$

$$(5) \quad \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} = \sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} + S(n) \frac{\rho_n}{\rho_n + \nu} + \sum_{k=n+1}^{\infty} S(k) \left[ \frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right].$$

For the first two terms on the right side of (5) we have

$$(6) \quad \sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} + S(n) \frac{\rho_n}{\rho_n + \nu} \leq \frac{2S(0)\rho_n}{\nu} < \frac{2S(0)\rho_n}{\nu^\alpha}.$$

To estimate the third term consider first

$$(7) \quad \left[ \frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right] = \frac{\nu(\rho_k - \rho_{k-1})}{(\rho_k + \nu)(\rho_{k-1} + \nu)}.$$

Since  $\rho_k - \rho_{k-1} = 2(\pi/t)^2 k$  and  $(\rho_k + \nu)(\rho_{k-1} + \nu) \geq [(\pi k/2t)^2 + \nu]^2$ , the right side of (7) is dominated by  $2(\pi/t)^2 \nu k [(\pi k/2t)^2 + \nu]^{-2}$ . Applying (4) and the above, we have

$$(8) \quad \sum_{k=n+1}^{\infty} S(k) \left[ \frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right] \leq 2(\pi/t)^2 A \nu \sum_{k=n+1}^{\infty} \frac{k^{1-2\alpha}}{[(\pi k/2t)^2 + \nu]^2}.$$

To get the desired estimate we will use standard integral estimates. For  $\alpha \geq \frac{1}{2}$ , the summands in the right side of (8) decrease monotonically with  $k$  for fixed  $\nu$ . If  $\alpha < \frac{1}{2}$ , the function

$$g(\xi) = \xi^{1-2\alpha} \left[ \left( \frac{\pi \xi}{2t} \right)^2 + \nu \right]^{-2}, \quad \xi \geq 0$$

has a unique local and absolute maximum at

$$\xi^* = \frac{2t}{\pi} \left( \frac{1 - 2\alpha}{3 + 2\alpha} \nu \right)^{1/2}.$$

In this case if  $n \geq \xi^*$ , the summands in the right side of (8) decrease monotonically as  $k$  increases and

$$\begin{aligned} (9) \quad 2 \left( \frac{\pi}{t} \right)^2 A \nu \sum_{k=n+1}^{\infty} \frac{k^{1-2\alpha}}{[(\pi k/2t)^2 + \nu]^2} &\leq \frac{2(\pi/t)^2 A}{\nu} \int_n^{\infty} \frac{\xi^{1-2\alpha}}{[(\pi \xi/2t \sqrt{\nu})^2 + 1]^2} d\xi \\ &= \frac{8(\pi/2t)^{2\alpha} A}{\nu^\alpha} \int_{\pi n/2t \sqrt{\nu}}^{\infty} \frac{\eta^{1-2\alpha}}{(\eta^2 + 1)^2} d\eta \\ &< 8 \left( \frac{\pi}{2t} \right)^{2\alpha} A \int_0^{\infty} \frac{\eta^{1-2\alpha}}{(\eta^2 + 1)^2} d\eta \frac{1}{\nu^\alpha}. \end{aligned}$$

In the case  $\alpha \geq \frac{1}{2}$ , (9) holds for any  $n$  and in both cases the last integral converges since  $0 < \alpha < 1$ . To complete the proof we fix in (5)  $n = n^* \geq \xi^*$  in the case  $\alpha < \frac{1}{2}$  or  $n = n^* \geq 1$  if  $\alpha \geq \frac{1}{2}$ . Estimates (6), (8), and (9) complete the proof.

REFERENCES

1. N. I. Achiezer, *Theory of Approximation* (Ungar, New York, 1956).
2. R. H. Cameron and M. D. Donsker, *Inversion formulae for characteristic functionals of stochastic processes*, Ann. of Math., **69** (1959) 15-36.

Received September 25, 1970. This work is contained in the author's doctoral dissertation done under the direction of Professor M. D. Donsker at the Courant Institute of Mathematical Sciences, New York University.

SAN JOSE STATE COLLEGE



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON  
Stanford University  
Stanford, California 94305

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

C. R. HOBBY  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

# Pacific Journal of Mathematics

Vol. 37, No. 3

March, 1971

Mohammad Shafqat Ali and Marvin David Marcus, <i>On the degree of the minimal polynomial of a commutator operator</i> . . . . .	561
Howard Anton and William J. Pervin, <i>Integration on topological semifields</i> . . . . .	567
Martin Bartelt, <i>Multipliers and operator algebras on bounded analytic functions</i> . . . . .	575
Donald Earl Bennett, <i>Aposyndetic properties of unicoherent continua</i> . . . . .	585
James W. Bond, <i>Lie algebras of genus one and genus two</i> . . . . .	591
Mario Borelli, <i>The cohomology of divisorial varieties</i> . . . . .	617
Carlos R. Borges, <i>How to recognize homeomorphisms and isometries</i> . . . . .	625
J. C. Breckenridge, <i>Burkill-Cesari integrals of quasi additive interval functions</i> . . . . .	635
J. Csima, <i>A class of counterexamples on permanents</i> . . . . .	655
Carl Hanson Fitzgerald, <i>Conformal mappings onto <math>\omega</math>-swirly domains</i> . . . . .	657
Newcomb Greenleaf, <i>Analytic sheaves on Klein surfaces</i> . . . . .	671
G. Goss and Giovanni Viglino, <i>C-compact and functionally compact spaces</i> . . . . .	677
Charles Lemuel Hagopian, <i>Arcwise connectivity of semi-aposyndetic plane continua</i> . . . . .	683
John Harris and Olga Higgins, <i>Prime generators with parabolic limits</i> . . . . .	687
David Michael Henry, <i>Stratifiable spaces, semi-stratifiable spaces, and their relation through mappings</i> . . . . .	697
Raymond D. Holmes, <i>On contractive semigroups of mappings</i> . . . . .	701
Joseph Edmund Kist and P. H. Maserick, <i>BV-functions on semilattices</i> . . . . .	711
Shūichirō Maeda, <i>On point-free parallelism and Wilcox lattices</i> . . . . .	725
Gary L. Musser, <i>Linear semiprime <math>(p; q)</math> radicals</i> . . . . .	749
William Charles Nemitz and Thomas Paul Whaley, <i>Varieties of implicative semilattices</i> . . . . .	759
Jaroslav Nešetřil, <i>A congruence theorem for asymmetric trees</i> . . . . .	771
Robert Anthony Nowlan, <i>A study of H-spaces via left translations</i> . . . . .	779
Gert Kjærgaard Pedersen, <i>Atomic and diffuse functionals on a <math>C^*</math>-algebra</i> . . . . .	795
Tilak Raj Prabhakar, <i>On the other set of the biorthogonal polynomials suggested by the Laguerre polynomials</i> . . . . .	801
Leland Edward Rogers, <i>Mutually aposyndetic products of chainable continua</i> . . . . .	805
Frederick Stern, <i>An estimate for Wiener integrals connected with squared error in a Fourier series approximation</i> . . . . .	813
Leonard Paul Sternbach, <i>On <math>k</math>-shrinking and <math>k</math>-boundedly complete basic sequences and quasi-reflexive spaces</i> . . . . .	817
Pak-Ken Wong, <i>Modular annihilator <math>A^*</math>-algebras</i> . . . . .	825