

Pacific Journal of Mathematics

**AN ESTIMATE FOR WIENER INTEGRALS CONNECTED WITH
SQUARED ERROR IN A FOURIER SERIES APPROXIMATION**

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If a function $x(\sigma)$, $0 \leq \sigma \leq t$, is in $\text{Lip-}\alpha$, $0 < \alpha < 1$, $x(0) = 0$ and if c_k ($k = 0, 1, 2, \dots$) are its Fourier coefficients with respect to the functions $\sqrt{2/t} \sin [\pi(k + \frac{1}{2})\sigma/t]$, then it is known [1, pp. 171-172] that

$$(1) \quad \sum_{k \geq n} c_k^2 \leq \frac{A}{(n + \frac{1}{2})^{2\alpha}}, \quad n \geq 0$$

where A is a positive number not depending on n . We will show a connection between this estimate and an estimate for Wiener integrals. Let $E_w\{ \}$ denote expectation on a Wiener process, that is, a Gaussian process with mean function zero, covariance function $\min(\sigma, \tau)$, $0 \leq \sigma, \tau \leq t$ and sample functions $z(\sigma)$ with $z(0) = 0$.

THEOREM: Let $x(\sigma)$ be in $C[0, t]$ and let c_k be the Fourier coefficients of $x(\sigma)$ with respect to the normalized eigenfunctions associated with $\min(\sigma, \tau)$. That is

$$c_k = \sqrt{\frac{2}{t}} \int_0^t x(\sigma) \sin [\pi(k + \frac{1}{2})\sigma/t] d\sigma.$$

Let $0 < \alpha < 1$. Then estimate (1) is a necessary and sufficient condition for the estimate

$$(2) \quad e^{-(B/2)\nu^{1-\alpha}} \leq \frac{E_w \left\{ e^{-(\nu/2)} \int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma \right\}}{E_w \left\{ e^{-(\nu/2)} \int_0^t z^2(\sigma) d\sigma \right\}}$$

for all positive ν , where B is a positive number not depending on ν .

Proof. From Cameron and Donsker's proof of a lemma [2, p. 27-28], we have that, for the case $\rho_k = [\pi(k + \frac{1}{2})/t]^2$, the right side of (2) equals

$$e^{-\nu/2} \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu}.$$

Hence estimate (2) holds if and only if

$$(3) \quad \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leq \frac{B}{\nu^\alpha}$$

for all positive ν . To prove that (2) implies (1) note that for each fixed value of ν , as $k \rightarrow \infty$, $[\rho_k | (\rho_k + \nu)] \uparrow 1$. Therefore for each n , by the remark and (3),

$$\frac{\rho_n}{\rho_n + \nu} \sum_{k \geq n} c_k^2 \leq \sum_{k \geq n} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leq \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leq \frac{B}{\nu^\alpha}$$

for all positive ν . Letting $\nu = \rho_n$ we have

$$\sum_{k \geq n} c_k^2 \leq \frac{2B}{[\pi(n + \frac{1}{2})/t]^{2\alpha}} = \frac{A}{(n + \frac{1}{2})^{2\alpha}}$$

which is estimate (1).

We now show that the latter estimate implies (3). Since the left side of (3) is bounded by $\sum_{k=0}^{\infty} c_k^2$, estimate (3) holds for $0 < \nu \leq 1$. Hence it suffices to prove (3) for $\nu > 1$. To simplify notation set

$$(4) \quad S(n) = \sum_{k \geq n} c_k^2 \leq \frac{A}{(n + \frac{1}{2})^{2\alpha}}$$

by hypothesis. For any $n \geq 1$

$$(5) \quad \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} = \sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} + S(n) \frac{\rho_n}{\rho_n + \nu} + \sum_{k=n+1}^{\infty} S(k) \left[\frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right].$$

For the first two terms on the right side of (5) we have

$$(6) \quad \sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} + S(n) \frac{\rho_n}{\rho_n + \nu} \leq \frac{2S(0)\rho_n}{\nu} < \frac{2S(0)\rho_n}{\nu^\alpha}.$$

To estimate the third term consider first

$$(7) \quad \left[\frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right] = \frac{\nu(\rho_k - \rho_{k-1})}{(\rho_k + \nu)(\rho_{k-1} + \nu)}.$$

Since $\rho_k - \rho_{k-1} = 2(\pi/t)^2 k$ and $(\rho_k + \nu)(\rho_{k-1} + \nu) \geq [(\pi k/2t)^2 + \nu]^2$, the right side of (7) is dominated by $2(\pi/t)^2 \nu k [(\pi k/2t)^2 + \nu]^{-2}$. Applying (4) and the above, we have

$$(8) \quad \sum_{k=n+1}^{\infty} S(k) \left[\frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right] \leq 2(\pi/t)^2 A \nu \sum_{k=n+1}^{\infty} \frac{k^{1-2\alpha}}{[(\pi k/2t)^2 + \nu]^2}.$$

To get the desired estimate we will use standard integral estimates. For $\alpha \geq \frac{1}{2}$, the summands in the right side of (8) decrease monotonically with k for fixed ν . If $\alpha < \frac{1}{2}$, the function

$$g(\xi) = \xi^{1-2\alpha} \left[\left(\frac{\pi \xi}{2t} \right)^2 + \nu \right]^{-2}. \quad \xi \geq 0$$

has a unique local and absolute maximum at

$$\xi^* = \frac{2t}{\pi} \left(\frac{1 - 2\alpha}{3 + 2\alpha} \nu \right)^{1/2}.$$

In this case if $n \geq \xi^*$, the summands in the right side of (8) decrease monotonically as k increases and

$$\begin{aligned} (9) \quad 2 \left(\frac{\pi}{t} \right)^2 A \nu \sum_{k=n+1}^{\infty} \frac{k^{1-2\alpha}}{[(\pi k/2t)^2 + \nu]^2} &\leq \frac{2(\pi/t)^2 A}{\nu} \int_n^{\infty} \frac{\xi^{1-2\alpha}}{[(\pi \xi/2t \sqrt{\nu})^2 + 1]^2} d\xi \\ &= \frac{8(\pi/2t)^{2\alpha} A}{\nu^\alpha} \int_{\pi n/2t \sqrt{\nu}}^{\infty} \frac{\eta^{1-2\alpha}}{(\eta^2 + 1)^2} d\eta \\ &< 8 \left(\frac{\pi}{2t} \right)^{2\alpha} A \int_0^{\infty} \frac{\eta^{1-2\alpha}}{(\eta^2 + 1)^2} d\eta \frac{1}{\nu^\alpha}. \end{aligned}$$

In the case $\alpha \geq \frac{1}{2}$, (9) holds for any n and in both cases the last integral converges since $0 < \alpha < 1$. To complete the proof we fix in (5) $n = n^* \geq \xi^*$ in the case $\alpha < \frac{1}{2}$ or $n = n^* \geq 1$ if $\alpha \geq \frac{1}{2}$. Estimates (6), (8), and (9) complete the proof.

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