ON $k$-SHRINKING AND $k$-BOUNDEDLY COMPLETE BASIC SEQUENCES AND QUASI-REFLEXIVE SPACES

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A Banach space $X$ is called quasi-reflexive (of order $n$) if $\text{codim}_{X^{**}} \pi(X) < +\infty$ ($\text{codim}_{X^{**}} \pi(X) = n$), where $\pi$ denotes the canonical embedding of $X$ into its second conjugate $X^{**}$. R. Herman and R. Whitley have shown that every quasi-reflexive space contains an infinite dimensional reflexive subspace. In this paper this result is extended by showing that if $X$ is quasi-reflexive of order $n$ and $0 \leq k \leq n$ then $X$ contains a subspace which is quasi-reflexive of order $k$.

1. Preliminaries. Throughout this paper $X$ will denote a Banach space, $X^*$ its first conjugate and $X^{**}$ its second conjugate.

The sequence $\{x_i\}$ in $X$ is said to be basic if $\{x_i\}$ is a basis for $[x_i]$ (where $[x_i]$ denotes the closed span of $\{x_i\}$). The sequence of functionals $\{f_i\}$ in $[x_i]^*$ defined by $f_i(x_i) = \delta_{ij}$ (where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$) are called the functionals biorthogonal to $\{x_i\}$. We will write $\{x_i, f_i\}$ is a basic sequence. It is well known [10] that the sequence $\{x_i\}$ in $X$, $x_i \neq 0$ ($i = 1, 2, \cdots$), is basic if and only if there exists $K > 0$ such that

$$(1) \quad \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{m} a_i x_i \right\|$$

for $1 \leq n \leq m < +\infty$ and any choice of scalars $a_1, a_2, \cdots, a_m$.

If $\{x_i\}$ is a basic sequence we call the sequence $\{z_n\}$, $z_n \neq 0$ ($n = 1, 2, \cdots$), a block basic sequence [1] of $\{x_i\}$ if there exists a sequence of scalars $(a_i)$ and $0 = p_1 < p_2 < \cdots$ such that $z_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i$. By (1), $\{z_n\}$ is a basic sequence.

If $A$ and $B$ are subspaces of $X$ we will write $A \oplus B$ to denote the direct sum of $A$ and $B$, when for each $x \in [A, B]$ (where $[A, B]$ denotes the closed span of $A \cup B$) there exists unique $\alpha \in A$, $\beta \in B$ such that $x = \alpha + \beta$. If $X = A \oplus B$ and $\dim B = n$ ($\dim B = +\infty$) we write $\text{codim}_X A = n$ ($\text{codim}_X A = +\infty$). We will also write $\text{codim}_X A = +\infty$ if $X$ has no subspace $B$ such that $X = A \oplus B$.

**Lemma 1.1.** If $X = [A, B]$ where $A$ and $B$ are closed subspaces of $X$ and if $\dim B = n$ and $A \cap B = 0$ then $\text{codim}_X A = n$ and $X = A \oplus B$.

I. Singer has shown [8]:

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Lemma 1.2. Let \( A \) be a closed subspace of \( X \).

1° The intersection of every \((n + 1)\)-dimensional subspace of \( X \) with \( A \) contains a nonzero element if and only if \( \text{codim}_X A \leq n \).

2° There exists an \( n \)-dimensional subspace of \( X \) whose intersection with \( A \) contains only the zero element if and only if \( \text{codim}_X A \geq n \).

2. \( k \)-shrinking and \( k \)-boundedly complete basic sequences.

Definition. A basic sequence \( \{x_i, f_i\} \) is \( k \)-shrinking if \( \text{codim}_{\{x_i\}} [f_i] = k \) [8].

We note that a basic sequence is \( 0 \)-shrinking if and only if it is shrinking [3].

Lemma 2.1. If \( \{x_i, f_i\} \) is a basic sequence and \( f \in [x_i]^* \), then \( f \in [f_i] \) if and only if \( \|f|_{[x_{n+1}, x_{n+2}, \ldots]}\| \to 0 \) as \( n \to \infty \) (where \( f|_{[x_{n+1}, x_{n+2}, \ldots]} \) denotes the functional \( f \) restricted to \([x_{n+1}, x_{n+2}, \ldots]\)).

The proof is in [8].

Lemma 2.2. If \( \{x_i, f_i\} \) is an \( n \)-shrinking basic sequence and \( \{z^i\} \) is a block basic sequence of \( \{x^i\} \) then \( \{z^i\} \) is \( k \)-shrinking for some \( k \leq n \).

Proof. Let \( \{h_i\} \) be the functionals biorthogonal to \( \{z_i\} \). Suppose \([z_i]^* \) contains an \((n + 1)\)-dimensional subspace, spanned by the linearly independent elements \( g_1, g_2, \ldots, g_{n+1} \), which intersects \([h_i]\) in only the zero element. Let \( g'_i \in [x_i]^* \) be such that \( g'_i|_{[z_i]} = g_i \) \((i = 1, 2, \ldots, n + 1)\). Then by Lemma 2.1 the \((n + 1)\)-dimensional subspace of \([x_i]^* \) spanned by \( \{g'_i: 1 \leq i \leq n + 1\} \) intersects \([f_i] \) in only the zero element. This contradicts Lemma 1.2, 2°. Hence by Lemma 1.2, 1°, \( \text{codim}_{\{z_i\}} [h_i] \leq n \). This completes the proof.

Theorem 2.3. If \( \{x_i, f_i\} \) is an \( n \)-shrinking basic sequence and \( 0 \leq k \leq n \) then there is a \( k \)-shrinking block basic sequence of \( \{x_i\} \).

To prove this theorem we need two lemmas.

Lemma 2.4. If \( \{x_i, f_i\} \) is a basic sequence and \( \{g_i: 1 \leq i \leq n\} \) is a linearly independent set in \([x_i]^* \) such that \( \{g_i: 1 \leq i \leq n\} \cap [f_i] = 0 \) then there is a \( \delta > 0 \) such that

\[
(2) \quad \left\| g_j |_{[x^i]_{i=m}^n} \cap \bigcap_{i \neq j} g_i^{-1} (0) \right\| > \delta
\]

for \( m = 1, 2, \ldots \) and \( j = 1, 2, \ldots, n \).
Proof. Without loss of generality let $j = n$. Let
\[ B_m = [f_1, \ldots, f_{m-1}, g_1, \ldots, g_{n-1}]^L. \]
From the isometry between $[x_i]^*/[g_1, g_2, \ldots, g_{n-1}, f_1, f_2, \ldots, f_{m-1}]$ and $B_m^*$ [9] we have
\[ \|g_n \| B_m \| = \text{dist} (g_n, [g_1, g_2, \ldots, g_{n-1}, f_1, f_2, \ldots, f_{m-1}) \]
\[ \geq \text{dist} (g_n, [g_1, g_2, \ldots, g_{n-1}, f_1, f_2, \ldots]) > \delta > 0 \]
for $m = 1, 2, \ldots$ and for some $\delta$ since $g_n \in [g_1, g_2, \ldots, g_{n-1}, f_1, f_2, \ldots]$.

Lemma 2.5. Let $\{x_i, f_i\}$ be a basic sequence and $\|x_i\| > \delta > 0$ for $i = 1, 2, \ldots$ for some $\delta$. If $f \in [x_i]^*$ and $\sum_{i=1}^{\infty} |f(x_i)| < +\infty$ then $\|f \| [x_{n+1}, x_{n+2}, \ldots] \| \to 0$ as $n \to \infty$.

Proof. Let $K$ satisfy (1) for the sequence $\{x_i\}$. Thus, since $|f_i(x)| < 2K\delta^{-1}$ where $\|x\| \leq 1$,
\[ \sup \left\{ \left| \frac{1}{i-m+1} \sum_{i=m+1}^{\infty} f_i(x)x_i \right| : x \in [x_{n+1}, x_{n+2}, \ldots], \|x\| \leq 1 \right\} \]
\[ \leq 2K\delta^{-1} \sum_{i=m+1}^{\infty} |f(x_i)|. \]

Proof of theorem. Since the basic sequence $\{x_i, f_i\}$ is $n$-shrinking there exists a linearly independent set $\{g_i : 1 \leq i \leq n\} \subseteq [x_i]^*$ such that
\[ (3) \quad [x_i]^* = [f_i] \oplus [g_i : 1 \leq i \leq n]. \]

By (2) in Lemma 2.4 we can construct a block basic sequence $\{y_i\}$ of $\{x_i\}$ with the following properties:
\[ (4) \quad \frac{1}{2} < \|y_i\| < \frac{3}{2}, i = 1, 2, \ldots, \]
\[ (5) \quad |g_i(y_{nq+i})| > \delta > 0 \quad \text{for some } \delta, \text{ for } i = 1, 2, \ldots, n \]
and $q = 1, 2, \ldots$, and
\[ (6) \quad |g_i(y_{nq+i})| < 1/2^q \quad \text{for } i \neq j. \]

Let $1 \leq k \leq n$ and let $\{z_i\}$ be a subsequence of $\{y_i\}$ consisting of the elements of the form $y_{nq+i}$ where $i = 1, 2, \ldots, k$ and $q = 1, 2, \ldots$. Let $\{h_i\}$ be the sequence of functionals biorthogonal to $\{z_i\}$. If $f \in [f_i]$ then, by Lemma 2.1, $f \| [z_i] \| [h_i]$. Let $g_j = g_j \| [z_i]$ $(j = 1, 2, \ldots, n)$. Since every functional in $[z_i]^*$ is the restriction of some functional in $[x_i]^*$ we conclude by (3) that
\[ (7) \quad [z_i]^* = [g_i', g_2', \ldots, g_n', h_1, h_2, \ldots]. \]
From Lemmas 2.1 and 2.5 and (4), (6) above it follows that $g_i' \in [h_i]$,
Assume there exist scalars $\alpha_i, \alpha_2, \ldots, \alpha_k$ and $h \in [h_i]$ such that $\sum_{i=1}^{k} \alpha_i g'_i = h$. Hence

$$\alpha_i g'_i = h - \sum_{i=2}^{k} \alpha_i g'_i.$$  

But by (5) and (4), $\|g'_i[y_{ap+1}: p \geq m]\| > \delta$ for $m = 1, 2, \ldots$. Also by (4), (6) and Lemma 2.1, $\|\tilde{h} - \sum_{i=2}^{k} \alpha_i g'_i[y_{ap+1}: p \geq m]\| \to 0$ as $m \to \infty$. Therefore $\alpha_i = 0$. Similarly $\alpha_i = 0$ for $i = 2, 3, \ldots, k$. Thus we have shown that the set $\{g'_i: 1 \leq i \leq k\}$ is linearly independent and $[g'_i: 1 \leq i \leq k] \cap [h_i] = 0$. Thus by (7) and Lemma 1.1 we have codim$_B[h_i] = k$ and hence $\{z_i\}$ is $k$-shrinking.

The case $k = 0$ follows from [1, Thm. 3, p. 154] and the fact that a quasi-reflexive space contains an infinite dimensional reflexive subspace [5].

**Definition.** Let $\{x_i\}$ be a basic sequence. We define two spaces of sequences $B(x_i)$ and $C(x_i)$ by

$$B(x_i) = \left\{(a_i x_i): \sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\| < +\infty \right\}$$

and

$$C(x_i) = \left\{(a_i x_i): \sum_{i=1}^{\infty} a_i x_i \text{ exists} \right\}.$$  

Define a norm on $B(x_i)$ and $C(x_i)$ by $\| (a_i x_i) \| = \sup_n \| \sum_{i=1}^{n} a_i x_i \|$. With this norm $B(x_i)$ and $C(x_i)$ are Banach spaces and $B(x_i) \supseteq C(x_i)$. We say $\{x_i\}$ is $k$-boundedly complete if codim$_B[C(x_i)] = k$ [8].

We note that a basic sequence $\{x_i\}$ is 0-boundedly complete if and only if $\{x_i\}$ is boundedly complete [3].

**Lemma 2.6.** If $\{x_i\}$ is an $n$-boundedly complete basic sequence and $\{z_i\}$ is a block basic sequence of $\{x_i\}$ then $\{z_i\}$ is $k$-boundedly complete for some $k \leq n$.

**Proof.** Assume $B(z_i)$ has an $(n + 1)$-dimensional subspace $W$ which intersects $C(z_i)$ in only the zero element. But then $\tilde{\psi}(W)$ would be an $(n + 1)$-dimensional subspace of $B(x_i)$ which intersects $C(x_i)$ in only the zero element, where $\tilde{\psi}$ denotes the natural embedding of $B(z_i)$ into $B(x_i)$ (i.e., $\tilde{\psi}(a_i z_i) = (b_i x_i)$ if for each $n$ there is an $m \geq n$ such that $\sum_{i=1}^{n} a_i z_i = \sum_{i=1}^{m} b_i x_i$). This contradicts Lemma 1.2.1$^\circ$. By Lemma 1.2.1$^\circ$, codim$_B[z_i] C(z_i) \leq n$.

**Theorem 2.7.** Let $\{x_i\}$ be an $n$-boundedly complete basic sequence for $n \geq 1$. Then for $k \in \{0, 1\}$ there is a block basic sequence $\{z_i\}$ of $\{x_i\}$ which is $k$-boundedly complete.
Proof. For the case $k = 1$, it is clearly sufficient to show that
\{x_i\} admits a $m$-boundedly complete block basic sequence for some $m$,
$1 \leq m < n$ whenever $n > 1$. Since \{x_i\} is not $O$-boundedly complete
there is an element $(a, x_i) \in B(x_i) - C(x_i)$. Hence there exists
$0 = p_1 < p_2 < \cdots$ and $\delta > 0$ such that if
\[
y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i \langle x_i, \|y_n\| > \delta \text{ for } n = 1, 2, \cdots.
\]
By Lemma 2.6 \{y_d\} is $m$-boundedly complete for some $m \leq n$.
Assume $m = n$. Then there exists
\[
\{(b_{ki}, y_i) : 1 \leq k \leq n - 1\} \subseteq B(y_i) - C(y_i)
\]
such that $B(y_i) = C(y_i) \oplus \{(b_{ki}, y_i) : 1 \leq k \leq n - 1\} \oplus \{y_d\}$. By (1) there
exists $M > 0$ such that $\|b_{ki}\| < M$ and thus $\|b_{ki}\| \leq M^{\delta^{-1}} (i = 1, 2, \cdots, 1 \leq k \leq n - 1)$. Hence there is an increasing sequence of
positive integers $(n_i)$ and $b_1, \cdots, b_{n-1}$ such that
$\lim_{i \to \infty} b_{kn_i} = b_k$ and $|b_k - b_{kn_i}| < 1/2^i (i = 1, 2, \cdots, 1 \leq k \leq n - 1)$. Let $c_{ki} = b_{ki} - b_i$ and
d_{ki} = c_{ki} - c_{ki}' where $c_{ki}' = c_{ki}$ for $j \in \{n_i\}$ and $c_{ki}' = 0$ for $j \notin \{n_i\}$. Then
\[(8) \quad B(y_i) = C(y_i) \oplus \{(d_{ki}, y_i) : 1 \leq k \leq n - 1\} \oplus \{y_d\}
\]
and $d_{kj} = 0$ for $j \notin \{n_i\}$. Let \{m_i\} be the sequence of positive integers complementary to \{n_i\}.
We will show that \{y_{m_i}\} is $(n - 1)$-boundedly complete. Let
$(e_{m_i}, y_{m_i}) \in B(y_{m_i})$. Therefore $(e_{m_i}, y_{m_i}) \in B(y_i)$ where $e_j = 0$ if $j \notin \{m_i\}$. Thus
by (8) there exist scalars $\alpha_i, \alpha_z, \cdots, \alpha_n$ and $(u_i, y_i) \in C(y_i)$ such that
\[(e_{m_i}, y_{m_i}) = (u_{m_i}, y_{m_i}) + \sum_{k=1}^{n-1} \alpha_k (d_{ki}, y_i) + \alpha_n (y_i).
\]
Thus we obtain $\alpha_n = 0$ and $u_j = 0$ for $j \in m_i$. Hence
\[(e_{m_i}, y_{m_i}) = (u_{m_i}, y_{m_i}) + \sum_{k=1}^{n-1} \alpha_k (d_{km_i}, y_{m_i}).
\]
Thus by Lemma 1.1, \{y_{m_i}\} is $(n - 1)$-boundedly complete.

The existence of a 0-boundedly complete block basic sequence
again follows from [1, Thm. 3, p. 54] and [5].

Lemma 2.8. Let the basic sequence \{x_i\} be 1-shrinking and
1-boundedly complete. Then there is a block basic sequence \{z_i\} of \{x_i\}
which is either 1-shrinking and 0-boundedly complete or 0-shrinking
and 1-boundedly complete.

Proof. Let \{y_i\} be the block basic sequence constructed as in
Theorem 2.7. Then \{y_i\} is 1-boundedly complete. If \{y_i\} is 0-shrinking
we are done. If not, then by Lemma 2.2, \( \{y_i\} \) is 1-shrinking. Thus by Lemma 2.1 there exists \( f \in [y_i]^* \) and \( 0 = p_1 < q_1 < p_2 < q_2 < \cdots \) such that

\[
(9) \quad \|f\|_{[y_i; p_\alpha \leq i \leq q_\alpha]} > \delta > 0, \text{ for some } \delta \text{ and } n = 1, 2, \cdots.
\]

As in the proof of Theorem 2.7, the subsequence \( \{z_i\} \) of \( \{y_i\} \), formed by those elements in \( [y_i; p_\alpha \leq i \leq q_\alpha] \) (\( n = 1, 2, \cdots \)) is 0-boundedly complete. But by (9) \( \{z_i\} \) is 1-shrinking.

For other results on \( k \)-shrinking and \( k \)-boundedly complete basic sequences see [4].

3. Quasi-reflexive spaces. We will write \( \text{Ord} (X) = n \) to mean \( X \) is quasi-reflexive of order \( n \).

Civin and Yood have shown [2]:

**Theorem 3.1.** If \( \text{Ord} (X) = n \) and \( Y \) is a closed subspace of \( X \) then \( Y \) and the quotient space \( X/Y \) are quasi-reflexive and \( \text{Ord} (X) = \text{Ord} (Y) + \text{Ord} (X/Y) \)

I. Singer has shown [8]:

**Theorem 3.2.** If \( \{x_i\} \) is a basic sequence then \( \text{Ord} ([x_i]) = n \) if and only if there exist natural numbers \( k_1 \) and \( k_2 \) such that \( \{x_i\} \) is \( k_1 \)-shrinking and \( k_2 \)-boundedly complete and \( n = k_1 + k_2 \).

**Theorem 3.3.** If \( \{x_i\} \) is a basic sequence and \( \text{Ord} ([x_i]) = n > 0 \) then there exist block basic sequences \( \{y_i\} \) and \( \{z_i\} \) of \( \{x_i\} \) such that \( \text{Ord} ([y_i]) = 1 \) and \( \text{Ord} ([z_i]) = 0 \).

**Proof.** The existence of \( \{z_i\} \) such that \( \text{Ord} ([z_i]) = 0 \) again follows from [1] and [5].

By Theorem 2.3 and Lemma 2.6 there exists a block basic sequence \( \{y_i\} \) of \( \{x_i\} \) which is 1-shrinking and \( k \)-boundedly complete for some \( k \leq n \). If \( k = 0 \) then \( \text{Ord} ([y_i]) = 1 \) by Theorem 3.2. If \( k > 0 \) there exists, by Lemma 2.6, a block basic sequence \( \{y'_i\} \) of \( \{y_i\} \) which is 1-boundedly complete. If \( \{y'_i\} \) is 0-shrinking we are done. If not then \( \{y'_i\} \) is 1-shrinking and we now apply Lemma 2.8 to complete the proof.

**Theorem 3.4.** Let \( \text{Ord} (X) = n > 0 \). There exists separable subspaces \( Y_0, Y_1, \cdots, Y_n \) of \( X \) such that \( \text{Ord} (Y_k) = k \) and \( Y_k \subseteq Y_{k+1} \) for \( k = 0, 1, \cdots, n - 1 \).
Proof. By [6, p. 546] a quasi-reflexive space of order $n$ contains a basic sequence $\{x_i\}$ which is $n$-shrinking. Thus $\{x_i\}$ is $0$-boundedly complete. Let $Y_n = [x_i]$. Thus by Theorem 2.2, there is a block basic sequence of $\{x_i\}$ which is $(n-1)$-shrinking and $0$-boundedly complete. Hence there exists $Y_{n-1}$ such that $\text{Ord}(Y_{n-1}) = n - 1$ and $Y_n \supseteq Y_{n-1}$. We construct $Y_{n-2}, Y_{n-3}, \ldots, Y_0$ similarly.

We note that we have also shown that each $Y_k$ has a basis.

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