

Pacific Journal of Mathematics

**ON k -SHRINKING AND k -BOUNDEDLY COMPLETE BASIC
SEQUENCES AND QUASI-REFLEXIVE SPACES**

LEONARD PAUL STERNBACH

ON k -SHRINKING AND k -BOUNDEDLY COMPLETE BASIC SEQUENCES AND QUASI-REFLEXIVE SPACES

L. STERNBACH

A Banach space X is called quasi-reflexive (of order n) if $\text{codim}_{X^{}}\pi(X) < +\infty$ ($\text{codim}_{X^{**}}\pi(X) = n$), where π denotes the canonical embedding of X into its second conjugate X^{**} . R. Herman and R. Whitley have shown that every quasi-reflexive space contains an infinite dimensional reflexive subspace. In this paper this result is extended by showing that if X is quasi-reflexive of order n and $0 \leq k \leq n$ then X contains a subspace which is quasi-reflexive of order k .**

1. Preliminaries. Throughout this paper X will denote a Banach space, X^* its first conjugate and X^{**} its second conjugate.

The sequence $\{x_i\}$ in X is said to be *basic* if $\{x_i\}$ is a basis for $[x_i]$ (where $[x_i]$ denotes the closed span of $\{x_i\}$). The sequence of functionals $\{f_i\}$ in $[x_i]^*$ defined by $f_i(x_j) = \delta_{ij}$ (where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$) are called the functionals biorthogonal to $\{x_i\}$. We will write $\{x_i, f_i\}$ is a basic sequence. It is well known [10] that the sequence $\{x_i\}$ in X , $x_i \neq 0$ ($i = 1, 2, \dots$), is basic if and only if there exists $K > 0$ such that

$$(1) \quad \left\| \sum_{i=1}^n a_i x_i \right\| \leq K \left\| \sum_{i=1}^m a_i x_i \right\|$$

for $1 \leq n \leq m < +\infty$ and any choice of scalars a_1, a_2, \dots, a_m .

If $\{x_i\}$ is a basic sequence we call the sequence $\{z_n\}$, $z_n \neq 0$ ($n = 1, 2, \dots$), a block basic sequence [1] of $\{x_i\}$ if there exists a sequence of scalars (α_i) and $0 = p_1 < p_2 < \dots$ such that $z_n = \sum_{i=p_{n-1}+1}^{p_n} \alpha_i x_i$. By (1), $\{z_n\}$ is a basic sequence.

If A and B are subspaces of X we will write $A \oplus B$ to denote the direct sum of A and B , when for each $x \in [A, B]$ (where $[A, B]$ denotes the closed span of $A \cup B$) there exists unique $\alpha \in A$, $\beta \in B$ such that $x = \alpha + \beta$. If $X = A \oplus B$ and $\dim B = n$ ($\dim B = +\infty$) we write $\text{codim}_X A = n$ ($\text{codim}_X A = +\infty$). We will also write $\text{codim}_X A = +\infty$ if X has no subspace B such that $X = A \oplus B$.

LEMMA 1.1. *If $X = [A, B]$ where A and B are closed subspaces of X and if $\dim B = n$ and $A \cap B = 0$ then $\text{codim}_X A = n$ and $X = A \oplus B$.*

I. Singer has shown [8]:

LEMMA 1.2. *Let A be a closed subspace of X .*

1° *The intersection of every $(n + 1)$ -dimensional subspace of X with A contains a nonzero element if and only if $\text{codim}_X A \leq n$.*

2° *There exists an n -dimensional subspace of X whose intersection with A contains only the zero element if and only if $\text{codim}_X A \geq n$.*

2. *k -shrinking and k -boundedly complete basic sequences.*

DEFINITION. A basic sequence $\{x_i, f_i\}$ is k -shrinking if $\text{codim}_{[x_i]^*} [f_i] = k$ [8].

We note that a basic sequence is 0-shrinking if and only if it is shrinking [3].

LEMMA 2.1. *If $\{x_i, f_i\}$ is a basic sequence and $f \in [x_i]^*$, then $f \in [f_i]$ if and only if $\|f| [x_{n+1}, x_{n+2}, \dots]\| \rightarrow 0$ as $n \rightarrow \infty$ (where $f| [x_{n+1}, x_{n+2}, \dots]$ denotes the functional f restricted to $[x_{n+1}, x_{n+2}, \dots]$).*

The proof is in [8].

LEMMA 2.2. *If $\{x_i, f_i\}$ is an n -shrinking basic sequence and $\{z_i\}$ is a block basic sequence of $\{x_i\}$ then $\{z_i\}$ is k -shrinking for some $k \leq n$.*

Proof. Let $\{h_i\}$ be the functionals biorthogonal to $\{z_i\}$. Suppose $[z_i]^*$ contains an $(n + 1)$ -dimensional subspace, spanned by the linearly independent elements g_1, g_2, \dots, g_{n+1} , which intersects $[h_i]$ in only the zero element. Let $g'_i \in [x_i]^*$ be such that $g'_i| [z_i] = g_i$ ($i = 1, 2, \dots, n + 1$). Then by Lemma 2.1 the $(n + 1)$ -dimensional subspace of $[x_i]^*$ spanned by $\{g'_i: 1 \leq i \leq n + 1\}$ intersects $[f_i]$ in only the zero element. This contradicts Lemma 1.2, 2°. Hence by Lemma 1.2, 1°, $\text{codim}_{[z_i]^*} [h_i] \leq n$. This completes the proof.

THEOREM 2.3. *If $\{x_i, f_i\}$ is an n -shrinking basic sequence and $0 \leq k \leq n$ then there is a k -shrinking block basic sequence of $\{x_i\}$.*

To prove this theorem we need two lemmas.

LEMMA 2.4. *If $\{x_i, f_i\}$ is a basic sequence and $\{g_i: 1 \leq i \leq n\}$ is a linearly independent set in $[x_i]^*$ such that $[g_i: 1 \leq i \leq n] \cap [f_i] = 0$ then there is a $\delta > 0$ such that*

$$(2) \quad \left\| g_j| [x_i]_{i=m}^\infty \cap \bigcap_{\substack{i=1 \\ i \neq j}}^n g_i^{-1}(0) \right\| > \delta$$

for $m = 1, 2, \dots$ and $j = 1, 2, \dots, n$.

Proof. Without loss of generality let $j = n$. Let

$$B_m = [f_1, \dots, f_{m-1}, g_1, \dots, g_{n-1}]^\perp.$$

From the isometry between $[x_i]^*/[g_1, g_2, \dots, g_{n-1}, f_1, f_2, \dots, f_{m-1}]$ and B_m^* [9] we have

$$\begin{aligned} \|g_n|B_m\| &= \text{dist}(g_n, [g_1, g_2, \dots, g_{n-1}, f_1, f_2, \dots, f_{m-1}]) \\ &\geq \text{dist}(g_n, [g_1, g_2, \dots, g_{n-1}, f_1, f_2, \dots]) > \delta > 0 \end{aligned}$$

for $m = 1, 2, \dots$ and for some δ since $g_n \notin [g_1, g_2, \dots, g_{n-1}, f_1, f_2, \dots]$.

LEMMA 2.5. *Let $\{x_i, f_i\}$ be a basic sequence and $\|x_i\| > \delta > 0$ ($i = 1, 2, \dots$) for some δ . If $f \in [x_i]^*$ and $\sum_{i=1}^\infty |f(x_i)| < +\infty$ then $\|f|[x_{n+1}, x_{n+2}, \dots]\| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let K satisfy (1) for the sequence $\{x_i\}$. Thus, since $|f_i(x)| < 2K\delta^{-1}$ where $\|x\| \leq 1$,

$$\begin{aligned} \sup \left\{ \left| f \left(\sum_{i=m+1}^\infty f_i(x)x_i \right) \right| : x \in [x_{n+1}, x_{n+2}, \dots], \|x\| \leq 1 \right\} \\ \leq 2K\delta^{-1} \sum_{i=m+1}^\infty |f(x_i)|. \end{aligned}$$

Proof of theorem. Since the basic sequence $\{x_i, f_i\}$ is n -shrinking there exists a linearly independent set $\{g_i: 1 \leq i \leq n\} \subseteq [x_i]^*$ such that

$$(3) \quad [x_i]^* = [f_i] \oplus [g_i: 1 \leq i \leq n].$$

By (2) in Lemma 2.4 we can construct a block basic sequence $\{y_i\}$ of $\{x_i\}$ with the following properties:

$$(4) \quad \frac{1}{2} < \|y_i\| < \frac{3}{2}, i = 1, 2, \dots,$$

$$(5) \quad |g_i(y_{nq+i})| > \delta > 0 \text{ for some } \delta, \text{ for } i = 1, 2, \dots, n$$

and $q = 1, 2, \dots$, and

$$(6) \quad |g_i(y_{nq+j})| < 1/2^q \text{ for } i \neq j.$$

Let $1 \leq k \leq n$ and let $\{z_i\}$ be a subsequence of $\{y_i\}$ consisting of the elements of the form y_{nq+i} where $i = 1, 2, \dots, k$ and $q = 1, 2, \dots$. Let $\{h_i\}$ be the sequence of functionals biorthogonal to $\{z_i\}$. If $f \in [f_i]$ then, by Lemma 2.1, $f|[z_i] \in [h_i]$. Let $g'_j = g_j|[z_i]$ ($j = 1, 2, \dots, n$). Since every functional in $[z_i]^*$ is the restriction of some functional in $[x_i]^*$ we conclude by (3) that

$$(7) \quad [z_i]^* = [g'_1, g'_2, \dots, g'_n, h_1, h_2, \dots].$$

From Lemmas 2.1 and 2.5 and (4), (6) above it follows that $g'_i \in [h_i]$,

$k + 1 \leq i \leq n$. Assume there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_k$ and $h \in [h_i]$ such that $\sum_{i=1}^k \alpha_i g'_i = h$. Hence

$$\alpha_1 g'_1 = h - \sum_{i=2}^k \alpha_i g'_i .$$

But by (5) and (4), $\|g'_1 | [y_{np+1}: p \geq m]\| > \frac{2}{3} \delta$ for $m = 1, 2, \dots$. Also by (4), (6) and Lemma 2.1, $\|(h - \sum_{i=2}^k \alpha_i g'_i) | [y_{np+1}: p \geq m]\| \rightarrow 0$ as $m \rightarrow \infty$. Therefore $\alpha_1 = 0$. Similarly $\alpha_i = 0$ for $i = 2, 3, \dots, k$. Thus we have shown that the set $\{g'_i: 1 \leq i \leq k\}$ is linearly independent and $[g'_i: 1 \leq i \leq k] \cap [h_i] = 0$. Thus by (7) and Lemma 1.1 we have $\text{codim}_{[z_i]^*} [h_i] = k$ and hence $\{z_i\}$ is k -shrinking.

The case $k = 0$ follows from [1, Thm. 3, p. 154] and the fact that a quasi-reflexive space contains an infinite dimensional reflexive subspace [5].

DEFINITION. Let $\{x_i\}$ be a basic sequence. We define two spaces of sequences $B(x_i)$ and $C(x_i)$ by

$$B(x_i) = \left\{ (a_i x_i): \sup_n \left\| \sum_{i=1}^n a_i x_i \right\| < +\infty \right\}$$

and

$$C(x_i) = \left\{ (a_i x_i): \sum_{i=1}^{\infty} a_i x_i \text{ exists} \right\} .$$

Define a norm on $B(x_i)$ and $C(x_i)$ by $\| (a_i x_i) \| = \sup_n \| \sum_{i=1}^n a_i x_i \|$. With this norm $B(x_i)$ and $C(x_i)$ are Banach spaces and $B(x_i) \supseteq C(x_i)$. We say $\{x_i\}$ is k -boundedly complete if $\text{codim}_{B(x_i)} C(x_i) = k$ [8].

We note that a basic sequence $\{x_i\}$ is 0-boundedly complete if and only if $\{x_i\}$ is boundedly complete [3].

LEMMA 2.6. *If $\{x_i\}$ is an n -boundedly complete basic sequence and $\{z_i\}$ is a block basic sequence of $\{x_i\}$ then $\{z_i\}$ is k -boundedly complete for some $k \leq n$.*

Proof. Assume $B(z_i)$ has an $(n + 1)$ -dimensional subspace W which intersects $C(z_i)$ in only the zero element. But then $\phi(W)$ would be an $(n + 1)$ -dimensional subspace of $B(x_i)$ which intersects $C(x_i)$ in only the zero element, where ϕ denotes the natural embedding of $B(z_i)$ into $B(x_i)$ (i.e., $\phi(a_i z_i) = (b_i x_i)$ if for each n there is a $m \geq n$ such that $\sum_{i=1}^n a_i z_i = \sum_{i=1}^m b_i x_i$). This contradicts Lemma 1.2.1°. By Lemma 1.2.1°, $\text{codim}_{B(z_i)} C(z_i) \leq n$.

THEOREM 2.7. *Let $\{x_i\}$ be an n -boundedly complete basic sequence for $n \geq 1$. Then for $k \in \{0, 1\}$ there is a block basic sequence $\{z_i\}$ of $\{x_i\}$ which is k -boundedly complete.*

Proof. For the case $k = 1$, it is clearly sufficient to show that $\{x_i\}$ admits a m -boundedly complete block basic sequence for some m , $1 \leq m < n$ whenever $n > 1$. Since $\{x_i\}$ is not 0-boundedly complete there is an element $(a_i x_i) \in B(x_i) - C(x_i)$. Hence there exists $0 = p_1 < p_2 < \dots$ and $\delta > 0$ such that if

$$y_n = \sum_{i=p_{n+1}}^{p_{n+1}} a_i x_i, \|y_n\| > \delta \text{ for } n = 1, 2, \dots .$$

By Lemma 2.6 $\{y_i\}$ is m -boundedly complete for some $m \leq n$. Assume $m = n$. Then there exists

$$\{(b_{ki} y_i) : 1 \leq k \leq n - 1\} \subseteq B(y_i) - C(y_i)$$

such that $B(y_i) = C(y_i) \oplus [(b_{ki} y_i) : 1 \leq k \leq n - 1] \oplus [(y_i)]$. By (1) there exists $M > 0$ such that $\|b_{ki} y_i\| < M$ and thus $|b_{ki}| \leq M\delta^{-1}$ ($i = 1, 2, \dots, 1 \leq k \leq n - 1$). Hence there is an increasing sequence of positive integers (n_i) and b_1, \dots, b_{n-1} such that $\lim_{i \rightarrow \infty} b_{kn_i} = b_k$ and $|b_k - b_{kn_i}| < 1/2^i$ ($i = 1, 2, \dots, 1 \leq k \leq n - 1$). Let $c_{ki} = b_{ki} - b_i$ and $d_{ki} = c_{ki} - c'_{ki}$ where $c'_{kj} = c_{kj}$ for $j \in \{n_i\}$ and $c'_{kj} = 0$ for $j \notin \{n_i\}$. Then $(c'_{ki} y_i) \in C(y_i)$ and

$$(8) \quad B(y_i) = C(y_i) \oplus [(d_{ki} y_i) : 1 \leq k \leq n - 1] \oplus [(y_i)]$$

and $d_{kj} = 0$ for $j \in \{n_i\}$. Let $\{m_i\}$ be the sequence of positive integers complementary to $\{n_i\}$.

We will show that $\{y_{m_i}\}$ is $(n - 1)$ -boundedly complete. Let $(e_{m_i} y_{m_i}) \in B(y_{m_i})$. Therefore $(e_i y_i) \in B(y_i)$ where $e_j = 0$ if $j \in \{m_i\}$. Thus by (8) there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ and $(u_i y_i) \in C(y_i)$ such that

$$(e_i y_i) = (u_i y_i) + \sum_{k=1}^{n-1} \alpha_k (d_{ki} y_i) + \alpha_n (y_i) .$$

Thus we obtain $\alpha_n = 0$ and $u_j = 0$ for $j \in m_i$. Hence

$$(e_{m_i} y_{m_i}) = (u_{m_i} y_{m_i}) + \sum_{k=1}^{n-1} \alpha_k (d_{km_i} y_{m_i}) .$$

Thus by Lemma 1.1, $\{y_{m_i}\}$ is $(n - 1)$ -boundedly complete.

The existence of a 0-boundedly complete block basic sequence again follows from [1, Thm. 3, p. 54] and [5].

LEMMA 2.8. *Let the basic sequence $\{x_i\}$ be 1-shrinking and 1-boundedly complete. Then there is a block basic sequence $\{z_i\}$ of $\{x_i\}$ which is either 1-shrinking and 0-boundedly complete or 0-shrinking and 1-boundedly complete.*

Proof. Let $\{y_i\}$ be the block basic sequence constructed as in Theorem 2.7. Then $\{y_i\}$ is 1-boundedly complete. If $\{y_i\}$ is 0-shrinking

we are done. If not, then by Lemma 2.2, $\{y_i\}$ is 1-shrinking. Thus by Lemma 2.1 there exists $f \in [y_i]^*$ and $0 = p_1 < q_1 < p_2 < q_2 < \dots$ such that

$$(9) \quad \|f|_{[y_i: p_n \leq i \leq q_n]}\| > \delta > 0, \text{ for some } \delta \text{ and } n = 1, 2, \dots.$$

As in the proof of Theorem 2.7, the subsequence $\{z_i\}$ of $\{y_i\}$, formed by those elements in $[y_i: p_n \leq i \leq q_n]$ ($n = 1, 2, \dots$) is 0-boundedly complete. But by (9) $\{z_i\}$ is 1-shrinking.

For other results on k -shrinking and k -boundedly complete basic sequences see [4].

3. Quasi-reflexive spaces. We will write $\text{Ord}(X) = n$ to mean X is quasi-reflexive of order n .

Civin and Yood have shown [2]:

THEOREM 3.1. *If $\text{Ord}(X) = n$ and Y is a closed subspace of X then Y and the quotient space X/Y are quasi-reflexive and $\text{Ord}(X) = \text{Ord}(Y) + \text{Ord}(X/Y)$*

I. Singer has shown [8]:

THEOREM 3.2. *If $\{x_i\}$ is a basic sequence then $\text{Ord}([x_i]) = n$ if and only if there exist natural numbers k_1 and k_2 such that $\{x_i\}$ is k_1 -shrinking and k_2 -boundedly complete and $n = k_1 + k_2$.*

THEOREM 3.3. *If $\{x_i\}$ is a basic sequence and $\text{Ord}([x_i]) = n > 0$ then there exist block basic sequences $\{y_i\}$ and $\{z_i\}$ of $\{x_i\}$ such that $\text{Ord}([y_i]) = 1$ and $\text{Ord}([z_i]) = 0$.*

Proof. The existence of $\{z_i\}$ such that $\text{Ord}([z_i]) = 0$ again follows from [1] and [5].

By Theorem 2.3 and Lemma 2.6 there exists a block basic sequence $\{y_i\}$ of $\{x_i\}$ which is 1-shrinking and k -boundedly complete for some $k \leq n$. If $k = 0$ then $\text{Ord}([y_i]) = 1$ by Theorem 3.2. If $k > 0$ there exists, by Lemma 2.6, a block basic sequence $\{y'_i\}$ of $\{y_i\}$ which is 1-boundedly complete. If $\{y'_i\}$ is 0-shrinking we are done. If not then $\{y'_i\}$ is 1-shrinking and we now apply Lemma 2.8 to complete the proof.

THEOREM 3.4. *Let $\text{Ord}(X) = n > 0$. There exists separable subspaces Y_0, Y_1, \dots, Y_n of X such that $\text{Ord}(Y_k) = k$ and $Y_k \subseteq Y_{k+1}$ for $k = 0, 1, \dots, n-1$.*

Proof. By [6, p. 546] a quasi-reflexive space of order n contains a basic sequence $\{x_i\}$ which is n -shrinking. Thus $\{x_i\}$ is 0-boundedly complete. Let $Y_n = [x_i]$. Thus by Theorem 2.2, there is a block basic sequence of $\{x_i\}$ which is $(n - 1)$ -shrinking and 0-boundedly complete. Hence there exists Y_{n-1} such that $\text{Ord}(Y_{n-1}) = n - 1$ and $Y_n \supseteq Y_{n-1}$. We construct $Y_{n-2}, Y_{n-3}, \dots, Y_0$ similarly.

We note that we have also shown that each Y_k has a basis.

The author would like to thank the referee for his helpful suggestions.

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Received September 2, 1969 and in revised form September 15, 1970.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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