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This paper is concerned with modular annihilator A^*-algebras. Let A be an A^*-algebra, B a maximal commutative *-subalgebra of A and X_B the carrier space of B. We show that the following statements are equivalent: (i) A is a modular annihilator algebra. (ii) Every X_B is discrete. (iii) Every B is a modular annihilator algebra. (iv) The spectrum of every hermitian element of A has no nonzero limit points.

Let A be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$, A^{**} the second conjugate space of A and π_A the canonical embedding of A into A^{**}. We show that A is a modular annihilator algebra if and only if $\pi_A(A)$ is a two-sided ideal of A^{**} (with the Arens product). This generalizes a recent result by B. J. Tomiuk and the author.

The theory of (left, right) modular annihilator algebras was developed in [20]. In a recent paper [4], Barnes has extended this study to semi-simple Banach algebras. He has proved an interesting result which says that if A is a semi-simple Banach algebra, then A is modular annihilator if and only if the spectrum of every element of A has no nonzero limit points (see [4; p. 516, Theorem 4.2]). In this paper, we show that a similar result holds for A^*-algebras.

2. **Notation and preliminaries.** Notation and definitions not explicitly given are taken from Rickart's book [15].

For any subset E of a Banach algebra A, let $L_A(E)$ and $R_A(E)$ denote the left and right annihilators of E in A, respectively. Then A is called a modular annihilator algebra if, for every maximal modular left ideal I and for every maximal modular right ideal J, we have $R_A(I) = (0)$ if and only if $I = A$ and $L_A(J) = (0)$ if and only if $J = A$. Let A be a semi-simple modular annihilator Banach algebra. Then every left (right) ideal of A contains a minimal idempotent (see [2; p. 569, Theorem 4.2]).

A Banach algebra with an involution $x \rightarrow x^*$ is called a Banach *-algebra. A Banach *-algebra A is called a B^*-algebra if the norm and the involution satisfy the condition $\|x^*x\| = \|x\|^2$ $(x \in A)$. If A is a Banach *-algebra on which there is defined a second norm $|\cdot|$, which satisfies, in addition to the multiplicative condition $|xy| \leqq |x|\,|y|$, the B^*-algebra condition $|x^*x| = |x|^2$, then A is called an A^*-algebra. The norm $|\cdot|$ is called an auxiliary norm. Let A be an A^*-algebra. Then the involution $x \rightarrow x^*$ in A is continuous with respect to the given norm and the auxiliary norm and every closed *-subalgebra of

A is semi-simple (see [15; p. 187, Theorem (4.1.15)] and [15; p. 188, Theorem (4.1.19)]).

Let A be a Banach algebra which is a subalgebra of a Banach algebra $\mathfrak{U}$. For each subset E of A, $\mathrm{cl}(E)$ (resp. $\mathrm{cl}_A(E)$) will denote the closure of E in A (resp. $\mathfrak{U}$).

Let A be a Banach algebra. For each element $x \in A$, let $Sp_A(X)$ denote the spectrum of x in A. If A is commutative, X_A will denote the carrier space of A and $C_0(X_A)$ the algebra of all complex-valued functions on X_A, which vanishes at infinity. If A is a commutative B^*-algebra, then $\hat{A} = C_0(X_A)$.

In this paper, all algebras and spaces under consideration are over the complex field C.

3. Characterizations of modular annihilator A*-algebras. Our first result, which is interesting in its own right, is useful in § 5.

THEOREM 3.1. *Let A be an A^*-algebra. Then the following statements are equivalent*:

(i) *A is a modular annihilator algebra.*

(ii) *The carrier space of every maximal commutative *-subalgebra of A is discrete.*

(iii) *Every maximal commutative *-subalgebra of A is a modular annihilator algebra.*

(iv) *The spectrum of every hermitian element of A has no nonzero limit points.*

Proof. (i) $\Rightarrow$ (iii). This follows immediately from [4; p. 517, Corollary].

(iii) $\Rightarrow$ (i). Let $|.|$ be the auxiliary norm on A. Assume $x = x^* \in A$ and let B be a maximal commutative *-subalgebra of A containing x. Then B has dense socle in $|.|$ by [5; p. 288, Theorem 3.3]. Since the socle of B is included in the socle of A, x is in the closure of the socle of A. It follows that A has dense socle in $|.|$. By [21; p. 376, Lemma 2.8], $|.|$ is a Q-norm on every maximal commutative *-subalgebra of A. Thus $|.|$ is a Q-norm on A by [5; p. 258, Lemma 1.2]. Therefore A is a modular annihilator algebra by [20; p. 41, Lemma 3.11].

(ii) $\Rightarrow$ (iv). Let x be a hermitian element in A and let B be a maximal commutative *-subalgebra of A containing x. By [15; p. 111, Theorem (3.1.6)],

$$Sp_B(x) - (0) \subset \{f(x): f \in X_B\} \subset Sp_B(x) .$$

We suppose, on the contrary, that $Sp_B(x)$ has a nonzero limit point $f_0(x)$, where $f_0 \in X_B$. Let $\{f_n\}$ be a sequence in X_B such that

$f_n(x) \to f_0(x)$ and $f_n(x)$ are distinct. Let $\varepsilon = \frac{1}{2}|f_0(x)|$. We may assume that $|f_n(x)| \geqq \varepsilon$ $(n = 1, 2, \cdots)$. For this given ε, there corresponds a compact subset $K \subset X_B$ such that $|f(x)| < \varepsilon$ for all $f \notin K$. Since X_B is discrete, K is finite. Hence $\{f_n\} \not\subset K$. But $|f_n(x)| \geqq \varepsilon$ for all n. This is a contradiction. Therefore $Sp_A(x) = Sp_B(x)$ has no nonzero limit points.

(iv) $\Rightarrow$ (iii). Let B be a maximal commutative *-subalgebra of A. For each $x \in B$, we can write $x = y + iz$ where y and z are hermitian elements in B. Since $\hat{y}$ and $\hat{z}$ have no nonzero limit points in their range, it follows that $\hat{z} = \hat{y} + i\hat{z}$ has the same property. Therefore by [4; p. 515, Theorem 4.1], B is a modular annihilator algebra.

(iii) $\Rightarrow$ (ii). Let B be a maximal commutative *-subalgebra of A. Then by [2; p. 569, Theorem 4.2(6)], X_B is discrete in the hull-kernal topology. Therefore X_B is discrete in the finer Gelfand topology. This completes the proof of the theorem.

Let B be a commutative Banach algebra with carrier space X_B. Then B is called completely regular provided, for every closed subset $F \subset X_B$ and $p \in X_B - F$, there exists $x \in B$ such that $F(x) = (0)$ and $p(x) = 1$. A commutative Banach algebra with discrete carrier space is completely regular.

COROLLARY 3.2. *Let A be an A^*-algebra which is a dense subalgebra of a B^*-algebra $\mathfrak{A}$. Then A is a modular annihilator algebra if and only if the following conditions are satisfied:*

(a) *$\mathfrak{A}$ is a dual algebra.*

(b) *For Every maximal commutative *-subalgebra B of A, B and* cl(B) *have the same carrier space.*

Proof. Suppose A is a modular annihilator algebra. By [5; p. 287, Lemma 2.6], $\mathfrak{A}$ has dense socle and therefore is a dual algebra (see [11; p. 222, Theorem 2.1]). This gives (a). By Theorem 3.1(ii), the carrier space of B is discrete. Therefore B is completely regular. Hence it follows from [15; p. 175, Theorem (3.7.5)] that cl(B) and B have the same carrier space. This proves (b).

Conversely, suppose conditions (a) and (b) hold. Since $\mathfrak{A}$ is dual, cl(B) has discrete carrier space. Therefore the carrier space of B is also discrete. Theorem 3.1 now shows that A is a modular annihilator algebra. This completes the proof.

A Banach *-algebra A is called symmetric provided every element of the form-x^*x is quasi-regular in A.

COROLLARY 3.3. *Let A be an A^*-algebra which is a dense subalgebra of a dual B^*-algebra $\mathfrak{A}$. Then A is a modular annihilator algebra if and only if A is symmetric.*

Proof. If A is a modular annihilator algebra, then by the proof of [15; p. 266, Theorem (4.10.11)], A is symmetric. Conversely suppose A is symmetric. Let B be a maximal commutative *-subalgebra of A. Then by [15; p. 233, Corollary (4.7.7)], B is a semi-simple symmetric algebra. Therefore B and $\mathrm{cl}(B)$ have the same carrier space (see [13; p. 219, Corollary]). It follows now from Corollary 3.2 that A is a modular annihilator algebra and the proof is complete.

4. The Arens products on A^{**}.

Let A be a Banach algebra, A^* and A^{**} the conjugate and second conjugate spaces of A, respectively. The two Arens products on A^{**} are defined in stages according to the following rules (see [1]). Let $x, y \in A$, $f \in A^*$, $F, G \in A^{**}$.

(a) Define $f \circ x$ by $(f \circ x)(y) = f(xy)$. Then $f \circ x \in A^*$.

(b) Define $G \circ f$ by $(G \circ f)(x) = G(f \circ x)$. Then $G \circ f \in A^*$.

(c) Define $F \circ G$ by $(F \circ G)(f) = F(G \circ f)$. Then $F \circ G \in A^{**}$.

A^{**} with the Arens product $\circ$ is denoted by $(A^{**}, \circ)$.

(a′) Define $x \circ' f$ by $(x \circ' f)(y) = f(yx)$. Then $x \circ' f \in A^*$.

(b′) Define $f \circ' F$ by $(f \circ' F)(x) = F(x \circ' f)$. Then $f \circ' F \in A^*$.

(c′) Define $F \circ' G$ by $(F \circ' G)(f) = G(f \circ' F)$. Then $F \circ G \in A^{**}$.

A^{**} with the Arens product $\circ'$ is denoted by $(A^{**}, \circ')$.

Each of these products extends the original multiplication on A when A is canonically embedded in A^{**}. In general, $\circ$ and $\circ'$ are distinct on A^{**}. If they coincide on A^{**}, then A is called Arens regular.

NOTATION. Let A be a Banach algebra. The mapping π_A will denote the canonical embedding of A into A^{**} in the rest of the paper.

LEMMA 4.1. *Let A be a Banach algebra and let B be a maximal commutative subalgebra of A. If $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$, then $\pi_B(B)$ is a two-sided ideal of $(B^{**}, \circ)$.*

Proof. This follows from the proof of (b) $\Rightarrow$ (a) in [19; p. 533, Theorem 5.1].

Let A be a B^*-algebra. Then A is Arens regular and A^{**} is a B^*-algebra under the Arens product (see [7; p. 869, Theorem 7.1] or [17; p. 192, Theorem 5]).

Lemma 4.2. *Let A be a B^*-algebra. Then A is a dual algebra if and only if $\pi_A(A)$ is a two-sided ideal of A^{**}.*

Proof. This is [19; p. 533, Theorem 5.1].

5. The Arens product and modular annihilator A*-algebras. Throughout this section, unless otherwise stated, A will be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$. The norm on A (resp. $\mathfrak{A}$) is denoted by $\|.\|$ (resp. $|.|$). We shall often use, without explicitly mentioning, the following fact: For every $x \in A$, $y \in \mathfrak{A}$, we have

$$\|xy\| \leqq k\|x\|\,|y| \text{ and } \|yx\| \leqq k\|x\|\,|y|\,, \tag{5.1}$$

where k is a constant (see [14; p. 18, Lemma 4]).

LEMMA 5.1. *Let A be commutative. If $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$, then A is a modular annihilator algebra.*

Proof. Let X_A be the carrier space of A. It follows easily from [20; p. 40, Lemma 3.8] that A and $\mathfrak{A}$ have the same carrier space. Therefore $\hat{\mathfrak{A}} = C_0(X_A)$. We show that X_A is discrete. Suppose this not so. Let $f \in X_A$ and let $\{f_t\}$ be a net in X_A such that $f_t \to f$ and $f_t \neq f$ for all t. Let E be the closed subspace of A^* spanned by the f_t. We claim that $f \notin E$. In fact, we assume $f \in E$. Choose $0 < \varepsilon < \|f\|/2k$, where $\|f\|$ denotes the norm of f in $\|.\|$ and k is a constant given in (5.1). Since $f \in E$, there exists $k_i \in C$ and $f_i \in \{f_t\}$ $(i = 1, 2, \cdots, n)$ such that

$$\left\|f - \sum_{i=1}^{n} k_i f_i\right\| < \varepsilon\,. \tag{5.2}$$

Since $\hat{\mathfrak{A}} = C_0(X_A)$, there exists $x_i \in \mathfrak{A}$ such that $|x_i| = 1$, $f(x_i) = 1$ and $f_i(x_i) = 0$ $(i = 1, 2, \cdots, n)$. Let $x \in A$ be such that $\|x\| \leqq 1$ and $|f(x)| \geqq \|f\|/2$. By (5.1), we have

$$\left\|\frac{1}{k}(xx_1 \cdots x_n)\right\| \leqq \|x\|\,|x_1| \cdots |x_n| \leqq 1\,. \tag{5.3}$$

Since $f_i(xx_1 \cdots x_n) = 0$ $(i = 1, 2, \cdots, n)$, it follows from (5.2) and (5.3) that

$$|f(xx_1 \cdots x_n)| < k\varepsilon < \|f\|/2\,. \tag{5.4}$$

But

$$|f(xx_1 \cdots x_n)| = |f(x)| \geqq \|f\|/2\,.$$

This is a contradiction to (5.4). Hence $f \notin E$. Therefore there exists an element $F \in A^{**}$ such that $F(E) = (0)$ and $F(f) \neq 0$. Choose $y \in A$ such that $f(y) \neq 0$. Then $(F \circ \pi_A(y))(f) = F(f)f(y) \neq 0$. Since $f_t \in E$,

$(F \circ \pi_A(y))(f_t) = F(f_t)f_t(y) = 0$ for all t. This contradicts the facts that $F \circ \pi_A(y) \in \pi_A(A)$ and $f_t \to f$ in X_A. Therefore X_A is discrete and so by Theorem 3.1, A is a modular annihilator algebra. This completes the proof.

In the following theorem, $(\mathfrak{A}^{**}, *)$ will denote the Arens product on $\mathfrak{A}^{**}$ and π the canonical mapping of $\mathfrak{A}$ into $\mathfrak{A}^{**}$.

THEOREM 5.2. *Let A be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$. Then the following statements are equivalent*:

(i) *A is a modular annihilator algebra.*

(ii) *$\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. By Corollary 3.2, $\mathfrak{A}$ is a dual algebra and so by Lemma 4.2, $\pi(\mathfrak{A})$ is a two-sided ideal of $(\mathfrak{A}^{**}, *)$. Let e be an idempotent of A. Since A is a two-sided ideal of $\mathfrak{A}$, $eA = e\mathfrak{A}$. For each $f \in A^*$, we define the linear functional $f.e$ on $\mathfrak{A}$ by

$$(f.e)(y) = f(ey) \quad (y \in \mathfrak{A}) .$$

Then by (5.1), $f.e \in \mathfrak{A}^*$. For each $x \in A$, let Φ be the mapping on $\pi(eA)$ into A^{**} given by

$$\Phi(\pi(ex))(f) = \pi(ex)(f.e) ,$$

for all $f \in A^*$. Then $\Phi(\pi(ex)) = \pi_A(ex)$ and so Φ is a one-one mapping of $\pi(eA)$ onto $\pi_A(eA)$. For each $g \in \mathfrak{A}^*$, let $g|A$ be the restriction of g to A. Since $|.| \leqq \beta ||.||$ for a constant β, $g|A \in A^*$. For every element $F \in A^{**}$, let $\tilde{F}$ be the linear functional on $\mathfrak{A}^*$ defined by

$$\tilde{F}(g) = F(g|A) \quad (g \in A^*) .$$

Then $\tilde{F} \in \mathfrak{A}^{**}$. Since $\pi(e) * \tilde{F} \in \pi(\mathfrak{A})$, it follows that $\pi(e) * \tilde{F} \in \pi(e\mathfrak{A}) = \pi(eA)$. Straightforward calculations show that $\Phi(\pi(e) * \tilde{F}) = \pi_A(e) \circ F$ and therefore we have

$$\pi_A(e) \circ F \in \pi_A(A) \quad (F \in A^{**}) . \tag{5.5}$$

Let $\{e_t\}$ be a maximal orthogonal family of hermitian minimal idempotents in $\mathfrak{A}$. It is easy to see that $\{e_t\} \subset A$. Let $x \in A$ and $F \in A^{**}$. Since $\mathfrak{A}$ is a dual algebra, by [14; p. 23, Lemma 6], $x = \sum_t xe_t$ in $|.|$. Hence only a countable number of $xe_t \neq 0$; denote those e_t's for which $xe_t \neq 0$ by $e_1, e_2, \cdots$. Let $x_n = \sum_{i=1}^n xe_i$ $(n = 1, 2, \cdots)$. It follows from (5.5) that

$$\pi_A(x_n) \circ F \in \pi_A(A) \qquad (n = 1, 2, \cdots) . \tag{5.6}$$

For each $f \in A^*$, we have

$$|(\pi_A(x_n) \circ F - \pi_A(x) \circ F)(f)| = |F(f \circ (x_n - x))|$$
$$\leqq ||F|| \, ||f \circ (x_n - x)|| \leqq k \, ||F|| \, ||f|| \, |x_n - x| \, .$$

Since $x_n \to x$ in $|.|$, we have $\pi_A(x_n) \circ F \to \pi_A(x) \circ F$ in $||.||$. It follows from (5.6) that $\pi_A(x) \circ F \in \pi_A(A)$. A similar argument shows that $F \circ \pi_A(x) \in \pi_A(A)$. Therefore $\pi_A(A)$ is a two-sided ideal of A^{**}. This proves (ii). (ii) $\Rightarrow$ (i). This follows immediately from Lemma 4.1, Lemma 5.1 and Theorem 3.1. The proof of the theorem is complete.

Let A be a modular annihilator B^*-algebra. It follows from [8; p. 48, Theorem (2.9.5)(iii)] that A is dual (also see [20; p. 42, Theorem 4.7]). Therefore the preceding theorem generalizes Lemma 4.2.

COROLLARY 5.3. *Let A and $\mathfrak{A}$ be as in Theorem* 5.2. *Then the following statements are equivalent*:

(i) *$\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$*

(ii) *$\pi(\mathfrak{A})$ is a two-sided ideal of $(\mathfrak{A}^{**}, *)$.*

Proof. This follows from Theorem 5.2, Corollary 3.2, Lemma 4.2 and [20; p. 40, Theorem 3.7].

THEOREM 5.4. *Let A be a reflexive A^*-algebra which is a dense two-sided ideal of a B^*-algebra $\mathfrak{A}$, then A is dual.*

Proof. Since A is reflexive, by Theorem 5.2 and Corollary 3.2, $\mathfrak{A}$ is a dual algebra and hence is w.c.c. Therefore by [14; p. 31, Theorem 17], A is a dual algebra. This completes the proof.

It is well-known that a proper H^*-algebra is dual. This fact also follows from Theorem 5.4, since a proper H^*-algebra satisfies the conditions of Theorem 5.4 (see [14; p. 31]).

Let H be a Hilbert space and $B(H)$ the algebra of all continuous linear operators on H into itself with the usual operator bound norm. Let $LC(H)$ be the algebra of all completely continuous operators on H and let $\tau c(H)$ be the trace-class on H.

THEOREM 5.5. *There exists a dual A^*-algebra A which is a dense two-sided ideal of a B^*-algebra such that A is Arens regular and $A^{**} = \pi_A(A) + R^{**}$, where $R^{**} \neq (0)$ is the radical of A^{**}.*

Proof. Let $\{H_\lambda\}$ be a family of Hilbert spaces such that at least one H_λ is infinite dimensional. Let $A = (\sum_\lambda \tau c(H_\lambda))_1$ be the L_1-direct sum of $\{\tau c(H_\lambda)\}$ and let $\mathfrak{A} = (\sum_\lambda LC(H_\lambda))_0$ be the $B^*(\infty)$-sum of $\{LC(H_\lambda)\}$.

Then A is a dual A^*-algebra which is a dense two-sided ideal of $\mathfrak{A}$ (see Theorem 9.2 in [18]). It is easy to verify that, as Banach spaces, A is isometrically isomorphic to $\mathfrak{A}^*$ and that in turn $\mathfrak{A}^{**}$ is isometrically isomorphic to the normed full direct sum $\sum_\lambda B(H_\lambda)$ of $\{B(H_\lambda)\}$. Let F be a bounded linear functional on A^*. Its restriction to $(\sum_\lambda LC(H_\lambda))_0$ $(\subset \sum_\lambda B(H_\lambda))$ determines an element $F_1 \in \pi_A(A)$. Let

$$M = \{E \in A^{**}: E(g) = 0 \text{ for all } g \in (\textstyle\sum_\lambda LC(H_\lambda))_0\} .$$

It is clear that $F - F_1 \in M$. Since $\pi_A(A) \neq A^{**}$, $M \neq (0)$.

Let t_λ be the trace operator on H_λ. For all $f = (f_\lambda) \in A^* = \sum_\lambda B(H_\lambda)$ and $x = (x_\lambda)$, $y = (y_\lambda) \in A$, by [16; p. 47, Theorem 2] we have

$$\begin{aligned}(f \circ x)(y) &= f(xy) = \textstyle\sum_\lambda f_\lambda(x_\lambda y_\lambda) = \sum_\lambda t_\lambda(x_\lambda y_\lambda f_\lambda)\\ &= \textstyle\sum_\lambda t_\lambda(y_\lambda f_\lambda x_\lambda) = \sum_\lambda (f_\lambda x_\lambda)(y_\lambda)\\ &= (fx)(y) .\end{aligned}$$

Since $fx \in (\sum_\lambda LC(H_\lambda))_0$, we have

$$(\pi_A(x) \circ E)(f) = E(f \circ x) = E(fx) = 0 ,$$

for all $f \in A^*$, $E \in M$ and $x \in A$. Since $\pi_A(A)$ is weakly dense in A^{**}, it follows from the weak continuity of left multiplication that $A^{**} \circ M = (0)$. Similarly we can show that $M \circ' A^{**} = (0)$. Since $\pi_A(x) \circ F = \pi_A(x) \circ' F$ and $F \circ \pi_A(x) = F \circ' \pi_A(x)$ for all $F \in A^{**}$, $x \in A$, we have

$$M \circ \pi_A(A) = \pi_A(A) \circ M = \pi_A(A) \circ' M = M \circ' \pi_A(A) = (0) .$$

Let $F, G \in A^{**}$ and write $F = F_1 + (F - F_1)$ and $G = G_1 + (G - G_1)$ with $F_1, G_1 \in \pi_A(A)$. Since $F - F_1$ and $G - G_1 \in M$, we have $F \circ G = F_1 \circ G_1 = F \circ' G$ and so A is Arens regular by definition. Since $A^{**} \circ M = M \circ A^{**} = (0)$, M is a two-sided ideal of A^{**}. Now it is clear that M is contained in the radical R^{**} of A^{**}. Since $R^{**} \cap \pi_A(A) = (0)$, we have $M = R^{**}$ and therefore $A^{**} = \pi_A(A) + R^{**}$. This completes the proof.

COROLLARY 5.6. *$(\sum_\lambda \tau c(H_\lambda))_1^{**}$ is a *-algebra.*

Proof. This follows from Theorem 5.5 and [17; p. 186, Theorem 1].

6. Unsolved questions. 1. Let H be a Hilbert space. For $1 \leqq p < \infty$, let C_p be the algebra given in [9; p. 1089]. Then C_p is an A^*-algebra which is a dense two-sided ideal of $LC(H)$. It is easy to show that for each $T \in C_p$, T is contained in the closure of TC_p in

C_p. Therefore by [14; p. 28, Lemma 8], C_p is a dual algebra (also see [3; pp. 10 – 11]). For $p = 2$, C_p is an H^*-algebra and therefore $C_2^{**} = C_2$. For $p \neq 2$ and $1 \leqq p < \infty$, is C_p Arens regular and is C_p^{**} semi-simple?

2. Let A be a dual A^*-algebra which is a dense two-sided ideal of a B^*-algebra. Is A Arens regular?

REMARK. We know that a dual A^*-algebra may not be Arens regular. Let A be the group algebra of an infinite compact abelian group. Then A is a dual A^*-algebra which is not an ideal of $\mathfrak{A}$, where $\mathfrak{A}$ is the completion of A in an auxiliary norm (see [14; p. 32]). By [7; p. 857, Theorem 3.14], A is not Arens regular.

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