

Pacific Journal of Mathematics

**BANACH ALGEBRAS WHICH ARE IDEALS IN A BANACH
ALGEBRA**

BRUCE ALAN BARNES

BANACH ALGEBRAS WHICH ARE IDEALS IN A BANACH ALGEBRA

BRUCE A. BARNES

In this paper Banach algebras A which are ideals in a Banach algebra B are studied. The main results concern the relationship between the norms of A and B and the relationship between the closed ideals of A and B .

There are many examples of Banach algebras in analysis which are ideals in another Banach algebra. When G is a locally compact group, then the Segal algebras which are studied in H. Reiter's book [7] are ideals in $L^1(G)$. J. Cigler considers more general Banach algebras which are ideals in $L^1(G)$ in [2]. In the theory of operators on a Hilbert space \mathcal{H} , the C_p algebras discussed in [4, pp. 1088-1119] are ideals in the algebra of compact operators on \mathcal{H} (C_1 is the ideal of trace class operators and C_2 the ideal of Hilbert-Schmidt operators). Also as we point out in §4, every full Hilbert algebra is a dense *-ideal in a B^* -algebra.

When A is a Banach algebra which is an ideal in a Banach algebra B , we consider the relationship between the algebras A and B . First we prove that the norms of A and B are related by certain inequalities. As a consequence, if B is semi-simple, then A is a left and right Banach module of B [Theorem 2.3]. Also in this case our results show that A is an abstract Segal algebra with respect to B as defined by J. T. Burnham in [1]. Secondly we relate the closed left and right ideals of A to those of B . Of special interest here is the case where A contains a bounded approximate identity of B [Theorem 3.4]. Finally in §4 we consider the special case where A is a *-ideal in a B^* -algebra B . The results of this section apply to full Hilbert algebras.

1. Preliminaries and notation. When B is any Banach algebra, we denote the Banach algebra norm on B by $\|\cdot\|_B$. If M is a closed left ideal in the Banach algebra B , $B - M = \{b + M \mid b \in B\}$ is the quotient module B modulo M . $B - M$ is normed by the norm

$$\|b + M\|'_B = \inf \{\|b - m\|_B \mid m \in M\}.$$

Throughout this paper A is a given Banach algebra. We always use the term "ideal" to mean two-sided ideal. A is usually an ideal in a Banach algebra B . In this case when E is a subset of A , $\text{cl}(E)$ is the closure of E in B .

At this point we prove a proposition of a purely algebraic nature which is useful in what follows.

PROPOSITION 1.1. *Assume that R is a ring and I is an ideal of R . Assume that M is a modular maximal left [right] ideal of R such that $I \not\subset M$. Then*

- (1) *I acts strictly irreducibly on $R - M$, and*
- (2) *$I \cap M$ is a maximal modular left [right] ideal of I .*

Proof. We prove (1) first. Assume $u \in R$ has the property $R(1-u) \subset M$. Let $K = \{b \in R \mid Ib \subset M\}$. K is a left ideal of R , $M \subset K$, and $u \notin K$ (if $u \in K$, $I \subset M$, a contradiction). Therefore $K = M$. It follows by the definition of K , that when $b \notin M$, $Ib + M$ properly contains M , and therefore $Ib + M = R$. This suffices to prove (1).

Now consider $I \cap M$. If $a \in I$ and $au \in M$, then $a \in I \cap M$. Therefore $I \cap M = \{a \in I \mid a(u + M) = 0 + M\}$. By (1) we can choose $v \in I$ such that $v(u + M) = u + M$. Then $I(1 - v) \subset I \cap M$ by the characterization of $I \cap M$ given above. Assume that $a \in I$, $a \notin I \cap M$. Given $b \in I$ we can choose $c \in I$ such that $b - ca \in M$ by (1). Then $b = ca + (b - ca) \in Ia + I \cap M$. Therefore $I = Ia + I \cap M$. Which proves (2).

2. The basic norm inequalities. In this section we assume that A is a subalgebra of a Banach algebra B . There is a close connection between certain inequalities involving $\|\cdot\|_A$ and $\|\cdot\|_B$ and the algebraic property that A is an ideal in some closed subalgebra of B . The next proposition has been noted by other authors.

PROPOSITION 2.1. *Assume that*

(1) *there exists $D > 0$ such that $D \|a\|_A \geq \|a\|_B$ for all $a \in A$, and*

(2) *there exists $C > 0$ such that $\|ab\|_A \leq C \max\{\|a\|_A \|b\|_B, \|a\|_B \|b\|_A\}$ for all $a, b \in A$.*

Then A is an ideal in $\text{cl}(A)$.

Proof. Assume that $a \in A$ and $b \in \text{cl}(A)$ are given. Choose $\{b_n\} \subset A$ such that $\|b_n - b\|_B \rightarrow 0$. Then $\|ab_n - ab_m\|_A \leq C \|a\|_A \|b_n - b_m\|_B$, so that $\{ab_n\}$ is Cauchy in A . Then there exists $c \in A$ such that $\|ab_n - c\|_A \rightarrow 0$. By (1) $\|ab_n - c\|_B \rightarrow 0$, and since $\|ab_n - ab\|_B \rightarrow 0$, we have $ab = c$. This proves that A is a right ideal of B . The proof that A is a left ideal of B is similar.

Together the next two results establish a converse to Proposition 2.1.

PROPOSITION 2.2. *Assume that A is a dense ideal in a semi-simple Banach algebra B . Then there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$.*

Proof. We prove that the embedding $(A, \|\cdot\|_A) \rightarrow (B, \|\cdot\|_B)$ is a closed, and hence continuous, map. Assume that $\{a_n\} \subset A$, $b \in B$, $\|a_n\|_A \rightarrow 0$ and $\|a_n - b\|_B \rightarrow 0$. Let M be a modular maximal left ideal of B with $A \not\subset M$, and let $u \in B$ have the property that $B(1-u) \subset M$. Given $a \in A$, let T_a act on $B - M$ by $T_a(b + M) = ab + M$. By Proposition 1.1 (1), $a \rightarrow T_a$ is a strictly irreducible representation of A on $B - M$. Let P be the kernel of this representation. P is a primitive ideal of A , and therefore P is closed in A . A/P is a Banach algebra with norm $\|a + P\|'_A$, $a \in A$. Given $a \in A$, define $S_{a+P}(b + M) = ab + M$, $b \in M$. Then $a + P \rightarrow S_{a+P}$ is a faithful strictly irreducible representation of A/P into the bounded operators on $B - M$. Then a theorem of B. E. Johnson [6, Theorem 1, p. 537] implies that $a + P \rightarrow S_{a+P}$ is a continuous map. Since $\|a_n + P\|'_A \rightarrow 0$, then $\|a_n u + M\|'_B = \|S_{a_n+P}(u + M)\|'_B \rightarrow 0$. Also $\|(a_n - b)(u + M)\|'_B \rightarrow 0$. It follows that $bu + M = 0$, and thus $b = bu + (b - bu) \in M$. Then b must be in every modular maximal left ideal of B , so that by the semi-simplicity of B , $b = 0$.

THEOREM 2.3. *Assume that A is an ideal in a Banach algebra B . Assume that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Then there exists $C > 0$ such that*

- (1) $\|ab\|_A \leq C\|a\|_A\|b\|_B$ for all $a \in A$, $b \in B$, and
- (2) $\|ab\|_A \leq C\|a\|_B\|b\|_A$ for all $a \in B$, $b \in A$.

Proof. We prove only (1). Let L_a , $a \in A$ be the operator mapping B into A given by $L_a(b) = ab$, $b \in B$. We prove that L_a is continuous by showing that L_a is a closed map from B into A . Assume that $\{b_n\} \subset B$, $c \in A$, and $\|b_n\|_B \rightarrow 0$, $\|L_a(b_n) - c\|_A \rightarrow 0$. Then $\|ab_n - c\|_A \rightarrow 0$, and since the A -norm dominates the B -norm, $\|ab_n - c\|_B \rightarrow 0$. Also $\|ab_n\|_B \rightarrow 0$, and therefore $c = 0$.

Now since L_a is continuous, for each $a \in A$ there exists $M_a > 0$ such that $\|ab\|_A \leq M_a\|b\|_B$, $b \in B$. Given $b \in B$, let R_b be the operator mapping A into A defined by $R_b(a) = ab$, $a \in A$. We prove that R_b is a closed, and hence continuous, map from A to A . Assume that $\{a_n\} \subset A$, $c \in A$, $\|a_n\|_A \rightarrow 0$, and $\|R_b(a_n) - c\|_A \rightarrow 0$. Then $\|a_n\|_B \rightarrow 0$ and $\|a_n b - c\|_B \rightarrow 0$. Thus $c = 0$. Therefore for each $b \in B$, R_b is a continuous operator. Set $|R_b| = \sup\{\|R_b(a)\|_A \mid a \in A, \|a\|_A \leq 1\}$. Let $\mathcal{S} = \{R_b \mid b \in B, \|b\|_B \leq 1\}$. We have that $\|R_b(a)\|_B \leq M_a$ for each $a \in A$ and $R_b \in \mathcal{S}$. Then by the Uniform Boundedness Theorem there exists $C > 0$ such that $|R_b| \leq C$ for all $R_b \in \mathcal{S}$. Thus

$$\frac{\|R_b(a)\|_A}{\|a\|_A} \leq C$$

for all $a \in A$, $a \neq 0$, and all $b \in B$, $\|b\|_B \leq 1$. Finally it follows that

$$\|ab\|_A \leq C\|a\|_A\|b\|_B$$

for all $a \in A$ and $b \in B$.

We remark that if A satisfies the hypotheses of Theorem 2.3, then by (1) and (2) A is a left and right Banach module over B ; see [5, Definition (32.14), p. 263].

3. Closed left and right ideals. Now assuming that A is an ideal of B , we relate the closed left and right ideals of A to those of B . The most comprehensive results in this direction are obtained when A has an approximate identity. However in the general case we do have the following theorem concerning modular closed left and right ideals of A .

THEOREM 3.1. *Assume that A is a dense ideal of a Banach algebra B and that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Let M be a closed modular left [right] ideal of A . Then $M = A \cap \text{cl}(M)$.*

Proof. By Theorem 2.3 there exists $C > 0$ such that $\|ab\|_A \leq C\|a\|_B\|b\|_A$ for all $a, b \in A$. Assume that M is a closed modular left ideal of A . Then there exists $u \in A$ such that $A(1-u) \subset M$. Given $a \in A$, $a = au + (a - au)$ and $a - au \in M$. Therefore $\|a + M\|'_A = \|au + M\|'_A$. Also $\|au + M\|'_A \leq \|au - bu\|_A$ for any $b \in M$ (note that when $b \in M$, then $bu \in M$). Therefore for all $b \in M$,

$$\|a + M\|'_A \leq \|au - bu\|_A \leq C\|a - b\|_B\|u\|_A.$$

Then $\|a + M\|'_A \leq (C\|u\|_A)\|a + M\|'_B$.

Assume that $a \in A \cap \text{cl}(M)$. Choose $\{a_n\} \subset M$ such that $\|a_n - a\|_B \rightarrow 0$. Then $\|(a_n - a) + M\|'_B \rightarrow 0$, and therefore $\|(a_n - a) + M\|'_A \rightarrow 0$. Thus there exists $\{b_n\} \subset M$ such that $\|(a_n - a) - b_n\|_A \rightarrow 0$. Since $\{a_n - b_n\} \subset M$, we have $a \in M$. Thus $A \cap \text{cl}(M) \subset M$. The opposite inclusion is immediate, so that $M = A \cap \text{cl}(M)$.

The next theorem provides a sufficient condition on A that every closed left [right] ideal of A is the intersection of A with a closed left [right] ideal of B . This theorem is proved by J. T. Burnham in [1, Theorem 1.1] (Theorem 2.3 removes one of Burnham's hypotheses).

THEOREM 3.2. *Assume that A is a dense ideal of B with the*

property that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Furthermore assume that for all $a \in A$, $a \in \overline{Aa}$ [$a \in \overline{aA}$] where “—” denotes closure in A . Then

(1) if N is a closed left [right] ideal of B , then $N \cap A$ is a closed left [right] ideal of A , and

(2) if M is a closed left [right] ideal of A , then $M = A \cap \text{cl}(M)$.

In many of the examples in harmonic analysis A is an ideal in $L^1(G)$ which contains a bounded approximate identity of $L^1(G)$. We prove that under these circumstances A has an approximate identity.

PROPOSITION 3.3. *Assume that A is a dense ideal in a Banach algebra B and that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Then if $\{e_\alpha\}$ is a left [right] bounded approximate identity for B and $\{e_\alpha\} \subset A$, $\{e_\alpha\}$ is a left [right] approximate identity for A .*

Proof. By Theorem 2.3 A is a left Banach module of B . Therefore by Cohen’s Theorem [5, Theorem (32.22), pp. [268] given $a \in A$ there exists $b \in B$ and $c \in A$ such that $a = bc$. Then

$$\|bc - e_\alpha bc\|_A \leq C\|b - e_\alpha b\|_B \|c\|_A \rightarrow 0.$$

Therefore $\{e_\alpha\}$ is a left approximate identity for A .

Combining several previous results, we have the following theorem which applies to many interesting examples in harmonic analysis.

THEOREM 3.4. *Assume that A is a dense ideal in a semi-simple Banach algebra B . Assume that A contains a bounded approximate identity of B . Then*

(1) for every closed left [right] ideal M of A , $M = A \cap \text{cl}(M)$, and

(2) if B has the property that every proper closed left [right] ideal of B is contained in a modular maximal left [right] ideal of B , then A has the property that every proper closed left [right] ideal of A is contained in a modular maximal left [right] ideal of A .

Proof. (1) follows from Proposition 2.2, Proposition 3.3, and Theorem 3.2. Then (1) and Proposition 1.1 imply (2).

4. **-ideals in a B -*algebra.* Assume that A is a full Hilbert algebra; see [9]. Then A is a pre-Hilbert space with the corresponding (linear) norm $\|\cdot\|_2$ on A . Also given $a \in A$, the operator U_a defined by $U_a(b) = ab$ for $b \in A$ is a bounded operator on $(A, \|\cdot\|_2)$. For $a \in A$ left $|a|$ denote the operator bound of U_a . Then $|\cdot|$ is an

algebra norm on A with the B^* -property. Let $\|a\|_A = \|a\|_2 + |a|$. M. Rieffel proves that $\|\cdot\|_A$ is a complete algebra norm on A [9, Proposition 1.15, p. 270]. Certainly $\|a\|_A \geq |a|$ for all $a \in A$. Also for all $a, b \in A$,

$$\begin{aligned} \|ab\|_A &= \|ab\|_2 + |ab| \\ &\leq |a| \|b\|_2 + |a| |b| \\ &\leq |a| \|b\|_A. \end{aligned}$$

Similarly $\|ab\|_A \leq \|a\|_A |b|$ for all $a, b \in A$. Let B be the completion of A in the norm $|\cdot|$. B is a B^* -algebra and A is a $*$ -subalgebra of B . Then by Proposition 2.1 A is a $*$ -ideal in B . Therefore every full Hilbert algebra is a $*$ -ideal in a B^* -algebra. In this section we consider briefly algebras A which are $*$ -ideals in B^* -algebras.

The next proposition is true in much more generality than we present here. When C is a Banach algebra, we denote the spectrum in C of an element $a \in C$ by $Sp_C(a)$. Also for $a \in C$ we let

$$\nu_C(a) = \inf(\|a^n\|_C^{1/n}).$$

PROPOSITION 4.1 *Assume that A is a dense $*$ -ideal in a semi-simple Banach $*$ -algebra B . Then every $*$ -representation of A on a Hilbert space \mathcal{H} extends uniquely to a $*$ -representation of B on \mathcal{H} .*

Proof. First note that by Johnson's Uniqueness of Norm Theorem [6, Theorem 2, p. 539] there exists $K > 0$ such that

$$\|b^*\|_B \leq K^2 \|b\|_B \text{ for all } b \in B.$$

Assume that $a \mapsto \pi(a)$ is a $*$ -representation of A into the bounded operators on a Hilbert space \mathcal{H} . If T is a bounded operator on \mathcal{H} , we denote the operator norm of T by $|T|$. By [8, Lemma (4.4.6), p. 208] $|\pi(a)|^2 \leq \nu_A(a^*a)$ for all $a \in A$. Since A is an ideal of B , then $Sp_A(a) \cup \{0\} = Sp_B(a) \cup \{0\}$ for all $a \in A$. Then $|\pi(a)|^2 \leq \nu_A(a^*a) = \nu_B(a^*a) \leq \|a^*a\|_B \leq K^2 \|a\|_B^2$ for all $a \in A$. Thus $|\pi(a)| \leq K \|a\|_B$ for all $a \in A$. Therefore π extends uniquely to a $*$ -representation of B on \mathcal{H} .

Now we prove the main result of this section.

THEOREM 4.2. *Assume that A is a dense $*$ -ideal in a B^* -algebra B . Then*

- (1) *A has an approximate identity consisting of self-adjoint elements.*
- (2) *For every closed left [right] ideal M of A , $M = A \cap \text{cl}(M)$.*
- (3) *Every proper closed left [right] ideal M of A in the inter-*

section of modular maximal left [right] ideals of A .

(4) Every $*$ -representation of A on a Hilbert space \mathcal{H} extends uniquely to a $*$ -representation of B on \mathcal{H} .

Proof. Construct the net $\{d_\lambda\}$, $\lambda \in A$, in A as in the proof of [8, Theorem (4.8.14), p. 245]. Then by this theorem and the fact that A is dense in B , $\{d_\lambda\}$, $\lambda \in A$, is a self-adjoint bounded approximate identity for B . Then by Proposition 3.3, $\{d_\lambda\}$, $\lambda \in A$, is an approximate identity for A . This proves (1). (2) follows from (1), Proposition 2.2, and Theorem 3.2.

Assume that M is a closed left ideal of A . Then by (2) $M = A \cap \text{cl}(M)$. By [3, Theorem 2.9.5, p. 48] $\text{cl}(M) = \bigcap_{\gamma \in \Gamma} N_\gamma$ where Γ is an index set and each N_γ is a modular maximal left ideal of B . By Proposition 1.1 $A \cap N_\gamma$ is a modular maximal left ideal of A for each $\gamma \in \Gamma$. Then $M = A \cap (\text{cl}(M)) = \bigcap_{\gamma \in \Gamma} (A \cap N_\gamma)$. This proves (3). Finally (4) follows from Proposition 4.1.

REFERENCES

1. J. T. Burnham, *Closed ideals in subalgebras of Banach algebras*; to appear.
2. J. Cigler, *Normed ideals in $L^1(G)$* , Nederl. Akad. Wetensch. Proc. Ser. A., **72** (1969), 273-282.
3. J. Dixmier, *Les C^* -algebres et Leurs Représentations*, Gautier-Villars, 1964.
4. N. Dunford and J. Schwartz, *Linear Operators, Part II*, Interscience Publishers, 1963.
5. E. Hewitt and K. Ross, *Abstract Harmonic Analysis II*, Springer-Verlage, 1970.
6. B. E. Johnson, *The uniqueness of the (complete) norm topology*, Bull. Amer. Math. Soc., **73** (1967), 573-539.
7. H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford, 1968.
8. C. Rickart, *Banach Algebras*, D. Van Nostrand, 1960.
9. M. Rieffel, *Square-integrable representations of Hilbert algebras*, J. Functional Analysis, **3** (1969), 265-300.

Received February 23, 1971. This research was partially supported by National Science Foundation Grant GP-20226.

UNIVERSITY OF OREGON

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
NAVAL WEAPONS CENTER

Pacific Journal of Mathematics

Vol. 38, No. 1

March, 1971

Bruce Alan Barnes, <i>Banach algebras which are ideals in a Banach algebra</i>	1
David W. Boyd, <i>Inequalities for positive integral operators</i>	9
Lawrence Gerald Brown, <i>Note on the open mapping theorem</i>	25
Stephen Daniel Comer, <i>Representations by algebras of sections over Boolean spaces</i>	29
John R. Edwards and Stanley G. Wayment, <i>On the nonequivalence of conservative Hausdorff methods and Hausdorff moment sequences</i>	39
P. D. T. A. Elliott, <i>On the limiting distribution of additive functions (mod 1)</i>	49
Mary Rodriguez Embry, <i>Classifying special operators by means of subsets associated with the numerical range</i>	61
Darald Joe Hartfiel, <i>Counterexamples to a conjecture of G. N. de Oliveira</i>	67
C. Ward Henson, <i>A family of countable homogeneous graphs</i>	69
Satoru Igari and Shigehiko Kuratsubo, <i>A sufficient condition for L^p-multipliers</i>	85
William A. Kirk, <i>Fixed point theorems for nonlinear nonexpansive and generalized contraction mappings</i>	89
Erwin Kleinfeld, <i>A generalization of commutative and associative rings</i>	95
D. B. Lahiri, <i>Some restricted partition functions. Congruences modulo 11</i>	103
T. Y. Lin, <i>Homological algebra of stable homotopy ring π_* of spheres</i>	117
Morris Marden, <i>A representation for the logarithmic derivative of a meromorphic function</i>	145
John Charles Nichols and James C. Smith, <i>Examples concerning sum properties for metric-dependent dimension functions</i>	151
Asit Baran Raha, <i>On completely Hausdorff-completion of a completely Hausdorff space</i>	161
M. Rajagopalan and Bertram Manuel Schreiber, <i>Ergodic automorphisms and affine transformations of locally compact groups</i>	167
N. V. Rao and Ashoke Kumar Roy, <i>Linear isometries of some function spaces</i>	177
William Francis Reynolds, <i>Blocks and F-class algebras of finite groups</i>	193
Richard Rochberg, <i>Which linear maps of the disk algebra are multiplicative</i>	207
Gary Sampson, <i>Sharp estimates of convolution transforms in terms of decreasing functions</i>	213
Stephen Scheinberg, <i>Fatou's lemma in normed linear spaces</i>	233
Ken Shaw, <i>Whittaker constants for entire functions of several complex variables</i>	239
James DeWitt Stein, <i>Two uniform boundedness theorems</i>	251
Li Pi Su, <i>Homomorphisms of near-rings of continuous functions</i>	261
Stephen Willard, <i>Functionally compact spaces, C-compact spaces and mappings of minimal Hausdorff spaces</i>	267
James Patrick Williams, <i>On the range of a derivation</i>	273