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**BANACH ALGEBRAS WHICH ARE IDEALS IN A BANACH
ALGEBRA**

BRUCE ALAN BARNES

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BRUCE A. BARNES

In this paper Banach algebras A which are ideals in a Banach algebra B are studied. The main results concern the relationship between the norms of A and B and the relationship between the closed ideals of A and B .

There are many examples of Banach algebras in analysis which are ideals in another Banach algebra. When G is a locally compact group, then the Segal algebras which are studied in H. Reiter's book [7] are ideals in $L^1(G)$. J. Cigler considers more general Banach algebras which are ideals in $L^1(G)$ in [2]. In the theory of operators on a Hilbert space \mathcal{H} , the C_p algebras discussed in [4, pp. 1088-1119] are ideals in the algebra of compact operators on \mathcal{H} (C_1 is the ideal of trace class operators and C_2 the ideal of Hilbert-Schmidt operators). Also as we point out in §4, every full Hilbert algebra is a dense *-ideal in a B^* -algebra.

When A is a Banach algebra which is an ideal in a Banach algebra B , we consider the relationship between the algebras A and B . First we prove that the norms of A and B are related by certain inequalities. As a consequence, if B is semi-simple, then A is a left and right Banach module of B [Theorem 2.3]. Also in this case our results show that A is an abstract Segal algebra with respect to B as defined by J. T. Burnham in [1]. Secondly we relate the closed left and right ideals of A to those of B . Of special interest here is the case where A contains a bounded approximate identity of B [Theorem 3.4]. Finally in §4 we consider the special case where A is a *-ideal in a B^* -algebra B . The results of this section apply to full Hilbert algebras.

1. Preliminaries and notation. When B is any Banach algebra, we denote the Banach algebra norm on B by $\|\cdot\|_B$. If M is a closed left ideal in the Banach algebra B , $B - M = \{b + M \mid b \in B\}$ is the quotient module B modulo M . $B - M$ is normed by the norm

$$\|b + M\|_B' = \inf \{\|b - m\|_B \mid m \in M\}.$$

Throughout this paper A is a given Banach algebra. We always use the term "ideal" to mean two-sided ideal. A is usually an ideal in a Banach algebra B . In this case when E is a subset of A , $\text{cl}(E)$ is the closure of E in B .

At this point we prove a proposition of a purely algebraic nature which is useful in what follows.

PROPOSITION 1.1. *Assume that R is a ring and I is an ideal of R . Assume that M is a modular maximal left [right] ideal of R such that $I \not\subset M$. Then*

- (1) I acts strictly irreducibly on $R - M$, and
- (2) $I \cap M$ is a maximal modular left [right] ideal of I .

Proof. We prove (1) first. Assume $u \in R$ has the property $R(1-u) \subset M$. Let $K = \{b \in R \mid Ib \subset M\}$. K is a left ideal of R , $M \subset K$, and $u \notin K$ (if $u \in K$, $I \subset M$, a contradiction). Therefore $K = M$. It follows by the definition of K , that when $b \notin M$, $Ib + M$ properly contains M , and therefore $Ib + M = R$. This suffices to prove (1).

Now consider $I \cap M$. If $a \in I$ and $au \in M$, then $a \in I \cap M$. Therefore $I \cap M = \{a \in I \mid a(u + M) = 0 + M\}$. By (1) we can choose $v \in I$ such that $v(u + M) = u + M$. Then $I(1 - v) \subset I \cap M$ by the characterization of $I \cap M$ given above. Assume that $a \in I$, $a \notin I \cap M$. Given $b \in I$ we can choose $c \in I$ such that $b - ca \in M$ by (1). Then $b = ca + (b - ca) \in Ia + I \cap M$. Therefore $I = Ia + I \cap M$. Which proves (2).

2. The basic norm inequalities. In this section we assume that A is a subalgebra of a Banach algebra B . There is a close connection between certain inequalities involving $\|\cdot\|_A$ and $\|\cdot\|_B$ and the algebraic property that A is an ideal in some closed subalgebra of B . The next proposition has been noted by other authors.

PROPOSITION 2.1. *Assume that*

(1) *there exists $D > 0$ such that $D \|a\|_A \geq \|a\|_B$ for all $a \in A$, and*

(2) *there exists $C > 0$ such that $\|ab\|_A \leq C \max\{\|a\|_A \|b\|_B, \|a\|_B \|b\|_A\}$ for all $a, b \in A$.*

Then A is an ideal in $\text{cl}(A)$.

Proof. Assume that $a \in A$ and $b \in \text{cl}(A)$ are given. Choose $\{b_n\} \subset A$ such that $\|b_n - b\|_B \rightarrow 0$. Then $\|ab_n - ab_m\|_A \leq C \|a\|_A \|b_n - b_m\|_B$, so that $\{ab_n\}$ is Cauchy in A . Then there exists $c \in A$ such that $\|ab_n - c\|_A \rightarrow 0$. By (1) $\|ab_n - c\|_B \rightarrow 0$, and since $\|ab_n - ab\|_B \rightarrow 0$, we have $ab = c$. This proves that A is a right ideal of B . The proof that A is a left ideal of B is similar.

Together the next two results establish a converse to Proposition 2.1.

PROPOSITION 2.2. *Assume that A is a dense ideal in a semi-simple Banach algebra B . Then there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$.*

Proof. We prove that the embedding $(A, \|\cdot\|_A) \rightarrow (B, \|\cdot\|_B)$ is a closed, and hence continuous, map. Assume that $\{a_n\} \subset A$, $b \in B$, $\|a_n\|_A \rightarrow 0$ and $\|a_n - b\|_B \rightarrow 0$. Let M be a modular maximal left ideal of B with $A \not\subset M$, and let $u \in B$ have the property that $B(1-u) \subset M$. Given $a \in A$, let T_a act on $B - M$ by $T_a(b + M) = ab + M$. By Proposition 1.1 (1), $a \rightarrow T_a$ is a strictly irreducible representation of A on $B - M$. Let P be the kernel of this representation. P is a primitive ideal of A , and therefore P is closed in A . A/P is a Banach algebra with norm $\|a + P\|_A'$, $a \in A$. Given $a \in A$, define $S_{a+P}(b + M) = ab + M$, $b \in M$. Then $a + P \rightarrow S_{a+P}$ is a faithful strictly irreducible representation of A/P into the bounded operators on $B - M$. Then a theorem of B. E. Johnson [6, Theorem 1, p. 537] implies that $a + P \rightarrow S_{a+P}$ is a continuous map. Since $\|a_n + P\|_A' \rightarrow 0$, then $\|a_n u + M\|_B' = \|S_{a_n+P}(u + M)\|_B' \rightarrow 0$. Also $\|(a_n - b)(u + M)\|_B' \rightarrow 0$. It follows that $bu + M = 0$, and thus $b = bu + (b - bu) \in M$. Then b must be in every modular maximal left ideal of B , so that by the semi-simplicity of B , $b = 0$.

THEOREM 2.3. *Assume that A is an ideal in a Banach algebra B . Assume that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Then there exists $C > 0$ such that*

- (1) $\|ab\|_A \leq C\|a\|_A \|b\|_B$ for all $a \in A$, $b \in B$, and
- (2) $\|ab\|_A \leq C\|a\|_B \|b\|_A$ for all $a \in B$, $b \in A$.

Proof. We prove only (1). Let L_a , $a \in A$ be the operator mapping B into A given by $L_a(b) = ab$, $b \in B$. We prove that L_a is continuous by showing that L_a is a closed map from B into A . Assume that $\{b_n\} \subset B$, $c \in A$, and $\|b_n\|_B \rightarrow 0$, $\|L_a(b_n) - c\|_A \rightarrow 0$. Then $\|ab_n - c\|_A \rightarrow 0$, and since the A -norm dominates the B -norm, $\|ab_n - c\|_B \rightarrow 0$. Also $\|ab_n\|_B \rightarrow 0$, and therefore $c = 0$.

Now since L_a is continuous, for each $a \in A$ there exists $M_a > 0$ such that $\|ab\|_A \leq M_a \|b\|_B$, $b \in B$. Given $b \in B$, let R_b be the operator mapping A into A defined by $R_b(a) = ab$, $a \in A$. We prove that R_b is a closed, and hence continuous, map from A to A . Assume that $\{a_n\} \subset A$, $c \in A$, $\|a_n\|_A \rightarrow 0$, and $\|R_b(a_n) - c\|_A \rightarrow 0$. Then $\|a_n\|_B \rightarrow 0$ and $\|a_n b - c\|_B \rightarrow 0$. Thus $c = 0$. Therefore for each $b \in B$, R_b is a continuous operator. Set $|R_b| = \sup\{\|R_b(a)\|_A \mid a \in A, \|a\|_A \leq 1\}$. Let $\mathcal{S} = \{R_b \mid b \in B, \|b\|_B \leq 1\}$. We have that $\|R_b(a)\|_B \leq M_a$ for each $a \in A$ and $R_b \in \mathcal{S}$. Then by the Uniform Boundedness Theorem there exists $C > 0$ such that $|R_b| \leq C$ for all $R_b \in \mathcal{S}$. Thus

$$\frac{\|R_b(a)\|_A}{\|a\|_A} \leq C$$

for all $a \in A$, $a \neq 0$, and all $b \in B$, $\|b\|_B \leq 1$. Finally it follows that

$$\|ab\|_A \leq C\|a\|_A\|b\|_B$$

for all $a \in A$ and $b \in B$.

We remark that if A satisfies the hypotheses of Theorem 2.3, then by (1) and (2) A is a left and right Banach module over B ; see [5, Definition (32.14), p. 263].

3. Closed left and right ideals. Now assuming that A is an ideal of B , we relate the closed left and right ideals of A to those of B . The most comprehensive results in this direction are obtained when A has an approximate identity. However in the general case we do have the following theorem concerning modular closed left and right ideals of A .

THEOREM 3.1. *Assume that A is a dense ideal of a Banach algebra B and that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Let M be a closed modular left [right] ideal of A . Then $M = A \cap \text{cl}(M)$.*

Proof. By Theorem 2.3 there exists $C > 0$ such that $\|ab\|_A \leq C\|a\|_B\|b\|_A$ for all $a, b \in A$. Assume that M is a closed modular left ideal of A . Then there exists $u \in A$ such that $A(1-u) \subset M$. Given $a \in A$, $a = au + (a - au)$ and $a - au \in M$. Therefore $\|a + M\|'_A = \|au + M\|'_A$. Also $\|au + M\|'_A \leq \|au - bu\|_A$ for any $b \in M$ (note that when $b \in M$, then $bu \in M$). Therefore for all $b \in M$,

$$\|a + M\|'_A \leq \|au - bu\|_A \leq C\|a - b\|_B\|u\|_A.$$

Then $\|a + M\|'_A \leq (C\|u\|_A)\|a + M\|'_B$.

Assume that $a \in A \cap \text{cl}(M)$. Choose $\{a_n\} \subset M$ such that $\|a_n - a\|_B \rightarrow 0$. Then $\|(a_n - a) + M\|'_B \rightarrow 0$, and therefore $\|(a_n - a) + M\|'_A \rightarrow 0$. Thus there exists $\{b_n\} \subset M$ such that $\|(a_n - a) - b_n\|_A \rightarrow 0$. Since $\{a_n - b_n\} \subset M$, we have $a \in M$. Thus $A \cap \text{cl}(M) \subset M$. The opposite inclusion is immediate, so that $M = A \cap \text{cl}(M)$.

The next theorem provides a sufficient condition on A that every closed left [right] ideal of A is the intersection of A with a closed left [right] ideal of B . This theorem is proved by J. T. Burnham in [1, Theorem 1.1] (Theorem 2.3 removes one of Burnham's hypotheses).

THEOREM 3.2. *Assume that A is a dense ideal of B with the*

property that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Furthermore assume that for all $a \in A$, $a \in \overline{Aa}$ [$a \in \overline{aA}$] where “—” denotes closure in A . Then

- (1) if N is a closed left [right] ideal of B , then $N \cap A$ is a closed left [right] ideal of A , and
- (2) if M is a closed left [right] ideal of A , then $M = A \cap \text{cl}(M)$.

In many of the examples in harmonic analysis A is an ideal in $L^1(G)$ which contains a bounded approximate identity of $L^1(G)$. We prove that under these circumstances A has an approximate identity.

PROPOSITION 3.3. *Assume that A is a dense ideal in a Banach algebra B and that there exists $D > 0$ such that $D\|a\|_A \geq \|a\|_B$ for all $a \in A$. Then if $\{e_\alpha\}$ is a left [right] bounded approximate identity for B and $\{e_\alpha\} \subset A$, $\{e_\alpha\}$ is a left [right] approximate identity for A .*

Proof. By Theorem 2.3 A is a left Banach module of B . Therefore by Cohen's Theorem [5, Theorem (32.22), pp. [268] given $a \in A$ there exists $b \in B$ and $c \in A$ such that $a = bc$. Then

$$\|bc - e_\alpha bc\|_A \leq C\|b - e_\alpha b\|_B \|c\|_A \rightarrow 0.$$

Therefore $\{e_\alpha\}$ is a left approximate identity for A .

Combining several previous results, we have the following theorem which applies to many interesting examples in harmonic analysis.

THEOREM 3.4. *Assume that A is a dense ideal in a semi-simple Banach algebra B . Assume that A contains a bounded approximate identity of B . Then*

- (1) for every closed left [right] ideal M of A , $M = A \cap \text{cl}(M)$, and
- (2) if B has the property that every proper closed left [right] ideal of B is contained in a modular maximal left [right] ideal of B , then A has the property that every proper closed left [right] ideal of A is contained in a modular maximal left [right] ideal of A .

Proof. (1) follows from Proposition 2.2, Proposition 3.3, and Theorem 3.2. Then (1) and Proposition 1.1 imply (2).

4. $*$ -ideals in a B - $*$ algebra. Assume that A is a full Hilbert algebra; see [9]. Then A is a pre-Hilbert space with the corresponding (linear) norm $\|\cdot\|_2$ on A . Also given $a \in A$, the operator U_a defined by $U_a(b) = ab$ for $b \in A$ is a bounded operator on $(A, \|\cdot\|_2)$. For $a \in A$ left $|a|$ denote the operator bound of U_a . Then $|\cdot|$ is an

algebra norm on A with the B^* -property. Let $\|a\|_A = \|a\|_2 + |a|$. M. Rieffel proves that $\|\cdot\|_A$ is a complete algebra norm on A [9, Proposition 1.15, p. 270]. Certainly $\|a\|_A \geq |a|$ for all $a \in A$. Also for all $a, b \in A$,

$$\begin{aligned} \|ab\|_A &= \|ab\|_2 + |ab| \\ &\leq |a| \|b\|_2 + |a| |b| \\ &\leq |a| \|b\|_A. \end{aligned}$$

Similarly $\|ab\|_A \leq \|a\|_A |b|$ for all $a, b \in A$. Let B be the completion of A in the norm $\|\cdot\|_A$. B is a B^* -algebra and A is a $*$ -subalgebra of B . Then by Proposition 2.1 A is a $*$ -ideal in B . Therefore every full Hilbert algebra is a $*$ -ideal in a B^* -algebra. In this section we consider briefly algebras A which are $*$ -ideals in B^* -algebras.

The next proposition is true in much more generality than we present here. When C is a Banach algebra, we denote the spectrum in C of an element $a \in C$ by $Sp_C(a)$. Also for $a \in C$ we let

$$\nu_C(a) = \inf(\|a^n\|_C^{1/n}).$$

PROPOSITION 4.1 *Assume that A is a dense $*$ -ideal in a semi-simple Banach $*$ -algebra B . Then every $*$ -representation of A on a Hilbert space \mathcal{H} extends uniquely to a $*$ -representation of B on \mathcal{H} .*

Proof. First note that by Johnson's Uniqueness of Norm Theorem [6, Theorem 2, p. 539] there exists $K > 0$ such that

$$\|b^*\|_B \leq K^2 \|b\|_B \text{ for all } b \in B.$$

Assume that $a \rightarrow \pi(a)$ is a $*$ -representation of A into the bounded operators on a Hilbert space \mathcal{H} . If T is a bounded operator on \mathcal{H} , we denote the operator norm of T by $|T|$. By [8, Lemma (4.4.6), p. 208] $|\pi(a)|^2 \leq \nu_A(a^*a)$ for all $a \in A$. Since A is an ideal of B , then $Sp_A(a) \cup \{0\} = Sp_B(a) \cup \{0\}$ for all $a \in A$. Then $|\pi(a)|^2 \leq \nu_A(a^*a) = \nu_B(a^*a) \leq \|a^*a\|_B \leq K^2 \|a\|_B^2$ for all $a \in A$. Thus $|\pi(a)| \leq K \|a\|_B$ for all $a \in A$. Therefore π extends uniquely to a $*$ -representation of B on \mathcal{H} .

Now we prove the main result of this section.

THEOREM 4.2. *Assume that A is a dense $*$ -ideal in a B^* -algebra B . Then*

- (1) *A has an approximate identity consisting of self-adjoint elements.*
- (2) *For every closed left [right] ideal M of A , $M = A \cap \text{cl}(M)$.*
- (3) *Every proper closed left [right] ideal M of A in the inter-*

section of modular maximal left [right] ideals of A .

(4) Every $*$ -representation of A on a Hilbert space \mathcal{H} extends uniquely to a $*$ -representation of B on \mathcal{H} .

Proof. Construct the net $\{d_\lambda\}$, $\lambda \in A$, in A as in the proof of [8, Theorem (4.8.14), p. 245]. Then by this theorem and the fact that A is dense in B , $\{d_\lambda\}$, $\lambda \in A$, is a self-adjoint bounded approximate identity for B . Then by Proposition 3.3, $\{d_\lambda\}$, $\lambda \in A$, is an approximate identity for A . This proves (1). (2) follows from (1), Proposition 2.2, and Theorem 3.2.

Assume that M is a closed left ideal of A . Then by (2) $M = A \cap \text{cl}(M)$. By [3, Theorem 2.9.5, p. 48] $\text{cl}(M) = \bigcap_{\gamma \in \Gamma} N_\gamma$ where Γ is an index set and each N_γ is a modular maximal left ideal of B . By Proposition 1.1 $A \cap N_\gamma$ is a modular maximal left ideal of A for each $\gamma \in \Gamma$. Then $M = A \cap (\text{cl}(M)) = \bigcap_{\gamma \in \Gamma} (A \cap N_\gamma)$. This proves (3). Finally (4) follows from Proposition 4.1.

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