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**CLASSIFYING SPECIAL OPERATORS BY MEANS OF  
SUBSETS ASSOCIATED WITH THE NUMERICAL RANGE**

MARY RODRIGUEZ EMBRY

## CLASSIFYING SPECIAL OPERATORS BY MEANS OF SUBSETS ASSOCIATED WITH THE NUMERICAL RANGE

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Let  $A$  be a continuous linear operator on a complex Hilbert space  $X$ , with inner product  $\langle, \rangle$  and associated norm  $\| \cdot \|$ . For each complex number  $z$  let  $M_z(A) = \{x: \langle Ax, x \rangle = z \|x\|^2\}$ . The following classifications of special operators are obtained: (i)  $A$  is a scalar multiple of an isometry if and only if  $AM_z(A) \subset M_z(A)$  for each complex  $z$ ; (ii)  $A$  is a nonzero scalar multiple of a unitary operator if and only if  $AM_z(A) = M_z(A)$  for each complex  $z$ ; and (iii)  $A$  is normal if and only if for each complex  $z$   $\{x \mid Ax \in M_z(A)\} = \{x \mid A^*x \in M_z(A)\}$ .

1. Introduction. The sets,  $M_z(A)$ , are closely associated with the numerical range of  $A$ :  $W(A) = \{\langle Ax, x \rangle: \|x\| = 1\}$ . These sets were introduced in [1] and used to characterize the elements of  $W(A)$  as follows:

THEOREM A. *If  $z \in W(A)$ , then*

- (i)  *$z$  is an extreme point of  $W(A)$  if and only if  $M_z(A)$  is linear,*
- (ii) *if  $z$  is a nonextreme boundary point of  $W(A)$ , then*

$$\gamma M_z(A) = \cup \{M_w(A): w \in L\}$$

where  $L$  is the line of support for  $W(A)$  passing through  $z$ ,

- (iii) *if  $W(A)$  is a convex body, then  $z$  is an interior point of  $W(A)$  if and only if  $\gamma M_z(A) = X$ .*

It was also shown in [1, Theorem 2] that  $\cap \{\text{maximal linear subspaces of } M_z(A)\}$  plays a special role in determining the normal eigenvalues of  $A$ .

With the aforementioned evidence concerning the sets  $M_z(A)$  in mind, it seemed natural to ask whether these sets behave in a particular fashion if  $A$  has special characteristics or whether the action of  $A$  on these sets determines special properties of  $A$ . Obviously  $A$  is Hermitian if and only if  $M_z(A) = M_{\bar{z}}(A)$  for all complex  $z$ . The first question which came to mind was: when is it the case that each of the sets  $M_z(A)$  is invariant under  $A$ . The techniques developed to answer this question in Theorem 1 led to the other theorems in this paper.

The following elementary facts can be noted about the sets,  $M_z(A)$ .

1. Each set  $M_z(A)$  is homogeneous and 2. either  $M_z(A) \cap M_w(A) = \{0\}$  or  $M_z(A) = M_w(A)$ .

2. **Notation and terminology.** The notation and terminology used in this paper are the same as that found in [1] with the following additions.  $f$  is a *bilinear functional* on a complex vector space  $X$  if and only if  $f: X \times X \rightarrow \{\text{complex numbers}\}$ ,  $f$  is linear in the first variable and conjugate linear in the second variable.

Throughout the paper  $A$  is a continuous linear operator on a complex Hilbert space  $X$ ;  $A$  is an *isometry* if  $A^*A = I$ ;  $A$  is *unitary* if  $A^*A = AA^* = I$ ;  $A$  is *normal* if  $AA^* = A^*A$ ; and  $A$  is *hyponormal* if  $AA^* \leq A^*A$ .  $\ker A$  denotes the null space of  $A$ :  $\{x: Ax = 0\}$ .

3. **Classification theorems.** The following lemma plays a fundamental part in the proofs of Theorems 1-4.

LEMMA 1. *If  $f, g, h$  and  $k$  are bilinear functionals on a complex vector space  $X$ , satisfying*

$$(1) \quad f(x, x)g(x, x) = h(x, x)k(x, x) \text{ for all } x \text{ in } X, \text{ then}$$

$$(2) \quad f(x, y)g(x, y) = h(x, y)k(x, y) \text{ for all } x \text{ and } y \text{ in } X.$$

*Indication of proof.* Let  $x, y \in X$  and let  $z$  be an arbitrary complex number. By substituting  $y + zx$  for  $x$  in equation (1) and equating coefficients, one arrives at equation (2) by means of the coefficients of  $z^2$ .

THEOREM 1.  *$A$  is a scalar multiple of an isometry if and only if  $AM_z(A) \subset M_z(A)$  for each complex  $z$ .*

*Proof.*  $M_z(A)$  is invariant under  $A$  for each complex  $z$  if and only if

$$(3) \quad \langle A^2x, Ax \rangle \|x\|^2 = \langle Ax, x \rangle \|Ax\|^2 \text{ for all } x \text{ in } X.$$

Obviously if  $A$  is a scalar multiple of an isometry, then equation (3) holds for all  $x$  in  $X$ . Thus we assume that equation (3) holds for all  $x$  in  $X$  and by Lemma 1 have

$$(4) \quad \langle A^2x, Ay \rangle \langle x, y \rangle = \langle Ax, y \rangle \langle Ax, Ay \rangle \text{ for all } x \text{ and } y \text{ in } X.$$

It now follows that  $\{x\}^\perp \subset \{Ax\}^\perp \cup \{A^*Ax\}^\perp$ . Moreover with  $x$  and  $y$  interchanged in (4) we see that  $\{x\}^\perp \subset \{A^*x\}^\perp \cup \{A^*Ax\}^\perp$ . Since  $\{y\}^\perp$

is linear, we have either  $\{x\}^\perp \subset \{A^*Ax\}^\perp$  or  $\{x\}^\perp \subset \{Ax\}^\perp \cap \{A^*x\}^\perp$ . Either case implies that there exists a scalar  $r_x$  such that  $A^*Ax = (r_x)x$ . This is sufficient to imply that  $A$  is a scalar multiple of an isometry.

If  $A$  is a nonunitary isometry, the only complex  $z$  in  $W(A)$  for which  $AM_z(A) = M_z(A)$  are the extreme points of  $W(A)$ . To prove this we make use of results from [2] and [3] which assert that in this case  $\sigma(A) = \overline{W(A)} = \{z: |z| \leq 1\}$ . Thus the elements of  $W(A)$  are either extreme points  $z$  with  $|z| = 1$  or interior points. If  $z$  is an extreme point of  $W(A)$ , then since  $A$  is hyponormal,

$$M_z(A) = \{x: Ax = zx \text{ and } A^*x = z^*x\}$$

by [4] and thus  $M_z(A) = AM_z(A) = A^*M_z(A)$ . Conversely if  $M_z(A) = AM_z(A)$ , then  $\gamma M_z(A) = A(\gamma M_z(A))$ . By Theorem A, (iii) if  $z$  is an interior point of  $W(A)$ , then  $X = AX$ , implying that  $A$  is invertible and hence unitary. Therefore if  $M_z(A) = AM_z(A)$  and  $z \in W(A)$ , then  $z$  is an extreme point of  $W(A)$ .

**THEOREM 2.**  *$A^*$  is a scalar multiple of an isometry if and only if  $A^*M_z(A) \subset M_z(A)$  for each complex  $z$ .*

*Proof.* Apply Theorem 1 to  $A^*$  and note that  $M_z(A^*) = M_{z^*}(A)$  for each complex  $z$ .

**THEOREM 3.**  *$A$  is a nonzero scalar multiple of a unitary operator if and only if  $AM_z(A) = M_z(A)$  for each complex  $z$ .*

*Proof.* By Theorems 1 and 2  $A$  is a scalar multiple of a unitary operator if and only if  $AM_z(A) \subset M_z(A)$  and  $A^*M_z(A) \subset M_z(A)$  for each complex  $z$ . Thus if  $A$  is nonzero, this is equivalent to  $AM_z(A) \subset M_z(A)$  and  $M_z(A) \subset AM_z(A)$ .

The proof of Theorem 4 which classifies normal operators in terms of the sets  $M_z(A)$  appears to depend upon the following lemma.

**LEMMA 2.** *If  $A$  and  $E$  are operators on  $X$  such that  $\ker A \subset \ker E$  and for each  $x$  in  $X$  either*

$$(i) \quad \|Ax\| = \|Ex\|$$

*or*

(ii) *there exists a real number  $r_x$  such that*

$$A^*Ax = (r_x)E^*Ex,$$

*then  $A^*A$  is a scalar multiple of  $E^*E$ .*

*Proof.* Assume that  $A^*Ax = aE^*Ex$  and  $A^*Ay = bE^*Ey$  where  $E^*Ex$  and  $E^*Ey$  are linearly independent. Let  $t$  be real,  $0 < t < 1$ . Either  $\|A(tx + (1-t)y)\| = \|E(tx + (1-t)y)\|$  or there exists a real number  $c$  such that  $A^*A(tx + (1-t)y) = cE^*E(tx + (1-t)y)$ . In this last case since  $0 < t < 1$  and  $E^*Ex$  and  $E^*Ey$  are linearly independent, we have  $a = c = b$ . Thus if  $a \neq b$ , then

$$\|A(tx + (1-t)y)\| = \|E(tx + (1-t)y)\|$$

for all  $t$ ,  $0 < t < 1$ . Letting  $t$  approach 1 and 0, we have  $\|Ax\| = \|Ex\|$  and  $\|Ay\| = \|Ey\|$ . Therefore  $|a| = |b| = 1$  and since  $E^*Ex \neq 0$  and  $E^*Ey \neq 0$ , necessarily  $a = b = 1$ . Thus we must have  $a = b$  if  $E^*Ex$  and  $E^*Ey$  are linearly independent.

Secondly if  $E^*Ex$  and  $E^*Ey$  are linearly dependent and  $A^*Ax = aE^*Ex$  and  $A^*Ay = bE^*Ey$ , then it follows from the hypothesis  $\ker A \subset \ker E$  that  $a$  and  $b$  can be chosen to be the same real number.

The arguments in the two preceding paragraphs show that there exists a real number  $r$  such that if  $x \in X$ , then either  $A^*Ax = rE^*Ex$  or  $\|Ax\| = \|Ex\|$ . Thus either  $\|Ax\| \leq \|Ex\|$  for all  $x$  in  $X$  or  $\|Ax\| \geq \|Ex\|$  for all  $x$  in  $X$ . In either case  $\{x: \|Ax\| = \|Ex\|\}$  is linear by Theorem A, (i). proving that  $X$  is the union of the two linear subspaces:

$$\{x: A^*Ax = rE^*Ex\} \quad \text{and} \quad \{x: \|Ax\| = \|Ex\|\}.$$

Therefore either  $A^*A = rE^*E$  or  $A^*A = E^*E$ .

**THEOREM 4.** *A is normal if and only if for each complex  $z$*

$$\{x | Ax \in M_z(A)\} = \{x | A^*x \in M_z(A)\}.$$

*Proof.* If  $A$  is normal it follows that  $Ax \in M_z(A)$  if and only if  $A^*x \in M_z(A)$ . Assume now that this condition holds. Then

$$(5) \quad \langle A^2x, Ax \rangle \|A^*x\|^2 = \langle AA^*x, A^*x \rangle \|Ax\|^2 \text{ for all } x \text{ in } X$$

and

$$(6) \quad \ker A = \ker A^*.$$

This last assertion can be proven as follows:  $x \in \ker A \leftrightarrow Ax \in M_z(A)$  for all complex  $z \leftrightarrow A^*x \in M_z(A)$  for all complex  $z \leftrightarrow x \in \ker A^*$ .

Using the same techniques as in the proof of Theorem 1, we show that if  $x \in X$ , either there exists a number  $b$  such that  $AA^*x = bA^*Ax$  or there exist numbers  $c$  and  $d$  such that  $AA^{*2}x = cAA^*x$  and  $A^*A^2x = dA^*Ax$ . These last two equations combined with (5) and (6) imply that either  $Ax = A^*x = 0$  or  $c = d^*$ . They also imply that  $A^{*2}x =$

$cA^*x$  and  $A^2x = dAx$ . Again using (6), we have  $AA^*x = cAx$  and  $A^*Ax = dA^*x$ . Thus if  $Ax \neq 0$ ,  $\|A^*x\|^2 = c \langle Ax, x \rangle = d^* \langle x, A^*x \rangle = \|Ax\|^2$ . Therefore  $A$  and  $A^*$  satisfy the hypotheses of Lemma 2 and there exists a real number  $r$  such that  $AA^* = rA^*A$ . This is sufficient to imply that  $A$  is normal.

**COROLLARY 5.** *Let  $A$  be an invertible operator on  $X$ . The following statements are equivalent:*

- (i)  $A$  is normal,
- (ii)  $A^{-1}M_z(A) = A^{*-1}M_z(A)$  for each complex  $z$ ,
- (iii)  $A^{-1}M_z(A^*A^{-1}) = A^{*-1}M_z(A^*A^{-1})$  for each complex  $z$ .

*Proof.* The equivalence of (i) and (ii) is a restatement of Theorem 4 for the case in which  $A$  is invertible. The equivalence of (i) and (iii) is obtained by applying Theorem 3 to the operator  $A^*A^{-1}$ .

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