COUNTEREXAMPLES TO A CONJECTURE OF G. N. DE OLIVEIRA

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G. N. de Oliveira gives the following conjecture.

CONJECTURE. Let \( A \) be an \( n \times n \) doubly stochastic irreducible matrix. If \( n \) is even, then \( f(z) = \text{perm}(Iz - A) \) has no real roots; if \( n \) is odd, then \( f(z) = \text{perm}(Iz - A) \) has one and only one real root.

In this paper we give counter examples to this conjecture.

Results:

EXAMPLE 1. Let

\[
A = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
0 & \frac{1}{4} & \frac{1}{2}
\end{bmatrix}.
\]

\( f(z) = \text{perm}(Iz - A) \) is such that \( f(0) < 0 \) and \( f(1) > 0 \). Consider \( f(z) \cdot (z - 1) = g(z) \). Note that \( g(0) > 0 \) and since there is a \( \xi(0 < \xi < 1) \) for which \( f(\xi) > 0 \) we see that \( g(\xi) < 0 \). Now consider

\[
B(\varepsilon) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{3}{4} - \varepsilon & \varepsilon \\
0 & 0 & \varepsilon & 1 - \varepsilon
\end{bmatrix}.
\]

If \( 0 \leq \varepsilon \leq \frac{1}{4} \), \( B(\varepsilon) \) is doubly stochastic. Further if \( g(z) = \text{perm}(Iz - B(\varepsilon)) \) then for each \( z \), \( g(z) = \lim_{\varepsilon \to 0} g_{(\varepsilon)}(z) \). Since \( g_{(\varepsilon)}(0) > 0 \) for each \( \varepsilon \) and \( g_{(\varepsilon)}(\xi) < 0 \) we see that for sufficiently small \( \varepsilon \), say \( \varepsilon_{0} \), \( g_{(\varepsilon_{0})}(z) \) has a real root and \( B(\varepsilon_{0}) \) is irreducible. This yields the counter-example. Note also that \( g_{(\varepsilon_{0})}(z) > 0 \) for \( z > 1 \) [see 1], hence \( g_{(\varepsilon_{0})}(z) \) has at least two real roots.

EXAMPLE 2. For simplification let \( B(\varepsilon_{0}) = B \) and \( g_{(\varepsilon_{0})}(z) = g(z) \).

Recall

(a) \( g(0) > 0 \) and
(b) \( g(\xi) < 0 \). By direct calculation we see that
(c) \( g(1) > 0 \) and hence for some \( \gamma, \xi < \gamma < 1 \)
(d) \( g(\gamma) > 0 \).

Now consider \( f(z) = g(z) \cdot (z - 1) \). Note that

(a) \( f(0) < 0 \)
(b) \( f(\xi) > 0 \)
(c) $f(1) = 0$
(d) $f(\gamma) < 0$.

Consider

$$A(\varepsilon) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & -\varepsilon_0 & \varepsilon_0 \\ 0 & 0 & \varepsilon_0 & 1 - \varepsilon_0 - \varepsilon & \varepsilon \\ 0 & 0 & 0 & \varepsilon & 1 - \varepsilon \end{bmatrix}$$

where $0 < \varepsilon < 1 - \varepsilon_0$.

Let $f_\varepsilon(z) = \text{perm} [Iz - A(\varepsilon)]$. Note that for each $z$, $\lim_{\varepsilon \to 0} f_\varepsilon(z) = f(z)$. Therefore for $\varepsilon$ sufficiently small, say $\varepsilon_i$

(a) $f_{\varepsilon_i}(0) < 0$
(b) $f_{\varepsilon_i}(z) > 0$
(c) $f_{\varepsilon_i}(\gamma) < 0$
(d) $f_{\varepsilon_i}(z) > 0$ for $z > 1$. Further $A(\varepsilon_i)$ is doubly stochastic and irreducible. Hence $f_{\varepsilon_i}(z)$ has at least three real roots. This yields a counter-example to the conjecture in the case $n$ is odd.

**References**


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