

# Pacific Journal of Mathematics

**COUNTEREXAMPLES TO A CONJECTURE OF G. N. DE  
OLIVEIRA**

DARALD JOE HARTFIEL

# COUNTEREXAMPLES TO A CONJECTURE OF G. N. DE OLIVEIRA

D. J. HARTFIEL

G. N. de Oliveira gives the following conjecture.

**CONJECTURE.** Let  $A$  be an  $n \times n$  doubly stochastic irreducible matrix. If  $n$  is even, then  $f(z) = \text{perm}(Iz - A)$  has no real roots; if  $n$  is odd, then  $f(z) = \text{perm}(Iz - A)$  has one and only one real root.

In this paper we give counter examples to this conjecture.

Results:

EXAMPLE 1. Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

$f(z) = \text{perm}(Iz - A)$  is such that  $f(0) < 0$  and  $f(1) > 0$ . Consider  $f(z) \cdot (z - 1) = g(z)$ . Note that  $g(0) > 0$  and since there is a  $\xi (0 < \xi < 1)$  for which  $f(\xi) > 0$  we see that  $g(\xi) < 0$ . Now consider

$$B(\varepsilon) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & -\varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & 1 - \varepsilon \end{bmatrix}.$$

If  $0 \leq \varepsilon \leq \frac{3}{4}$ ,  $B(\varepsilon)$  is doubly stochastic. Further if  $g_\varepsilon(z) = \text{perm}[Iz - B(\varepsilon)]$  then for each  $z$ ,  $g(z) = \lim_{\varepsilon \rightarrow 0} g_\varepsilon(z)$ . Since  $g_\varepsilon(0) > 0$  for each  $\varepsilon$  and  $g(\xi) = \lim_{\varepsilon \rightarrow 0} g_\varepsilon(\xi) < 0$  we see that for sufficiently small  $\varepsilon$ , say  $\varepsilon_0$ ,  $g_{\varepsilon_0}(z)$  has a real root and  $B(\varepsilon_0)$  is irreducible. This yields the counter-example. Note also that  $g_{\varepsilon_0}(z) > 0$  for  $z > 1$  [see 1], hence  $g_{\varepsilon_0}(z)$  has at least two real roots.

EXAMPLE 2. For simplification let  $B(\varepsilon_0) = B$  and  $g_{\varepsilon_0}(z) = g(z)$ . Recall

- (a)  $g(0) > 0$  and
- (b)  $g(\xi) < 0$ . By direct calculation we see that
- (c)  $g(1) > 0$  and hence for some  $\eta$ ,  $\xi < \eta < 1$
- (d)  $g(\eta) > 0$ .

Now consider  $f(z) = g(z) \cdot (z - 1)$ . Note that

- (a)  $f(0) < 0$
- (b)  $f(\xi) > 0$

- (c)  $f(1) = 0$   
 (d)  $f(\eta) < 0$ .

Consider

$$A(\varepsilon) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & -\varepsilon_0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 & 1 & -\varepsilon_0 & -\varepsilon & \varepsilon \\ 0 & 0 & 0 & \varepsilon & 0 & 1 & -\varepsilon \end{bmatrix}$$

where  $0 < \varepsilon < 1 - \varepsilon_0$ .

Let  $f_\varepsilon(z) = \text{perm}[Iz - A(\varepsilon)]$ . Note that for each  $z$ ,  $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(z) = f(z)$ . Therefore for  $\varepsilon$  sufficiently small, say  $\varepsilon_1$

- (a)  $f_{\varepsilon_1}(0) < 0$   
 (b)  $f_{\varepsilon_1}(\frac{1}{2}) > 0$   
 (c)  $f_{\varepsilon_1}(\eta) < 0$

(d)  $f_{\varepsilon_1}(z) > 0$  for  $z > 1$ . Further  $A(\varepsilon_1)$  is doubly stochastic and irreducible. Hence  $f_{\varepsilon_1}(z)$  has at least three real roots. This yields a counter-example to the conjecture in the case  $n$  is odd.

#### REFERENCES

1. R. A. Brualdi, and M. Newman, *Proof of a Permanent Inequality*, Quarterly Journal of Mathematics, Oxford (2), **17** (1966), 234-238.
2. G. N. De Oliveira, *A Conjecture and Some Problems on Permanents*, Pacific J. Math., **32**, No 2, (1970), 495-499.

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Bruce Alan Barnes, <i>Banach algebras which are ideals in a Banach algebra</i> . . . . .	1
David W. Boyd, <i>Inequalities for positive integral operators</i> . . . . .	9
Lawrence Gerald Brown, <i>Note on the open mapping theorem</i> . . . . .	25
Stephen Daniel Comer, <i>Representations by algebras of sections over Boolean spaces</i> . . . . .	29
John R. Edwards and Stanley G. Wayment, <i>On the nonequivalence of conservative Hausdorff methods and Hausdorff moment sequences</i> . . . . .	39
P. D. T. A. Elliott, <i>On the limiting distribution of additive functions (mod 1)</i> . . . . .	49
Mary Rodriguez Embry, <i>Classifying special operators by means of subsets associated with the numerical range</i> . . . . .	61
Darald Joe Hartfiel, <i>Counterexamples to a conjecture of G. N. de Oliveira</i> . . . . .	67
C. Ward Henson, <i>A family of countable homogeneous graphs</i> . . . . .	69
Satoru Igari and Shigehiko Kuratsubo, <i>A sufficient condition for <math>L^p</math>-multipliers</i> . . . . .	85
William A. Kirk, <i>Fixed point theorems for nonlinear nonexpansive and generalized contraction mappings</i> . . . . .	89
Erwin Kleinfeld, <i>A generalization of commutative and associative rings</i> . . . . .	95
D. B. Lahiri, <i>Some restricted partition functions. Congruences modulo 11</i> . . . . .	103
T. Y. Lin, <i>Homological algebra of stable homotopy ring <math>\pi_*</math> of spheres</i> . . . . .	117
Morris Marden, <i>A representation for the logarithmic derivative of a meromorphic function</i> . . . . .	145
John Charles Nichols and James C. Smith, <i>Examples concerning sum properties for metric-dependent dimension functions</i> . . . . .	151
Asit Baran Raha, <i>On completely Hausdorff-completion of a completely Hausdorff space</i> . . . . .	161
M. Rajagopalan and Bertram Manuel Schreiber, <i>Ergodic automorphisms and affine transformations of locally compact groups</i> . . . . .	167
N. V. Rao and Ashoke Kumar Roy, <i>Linear isometries of some function spaces</i> . . . . .	177
William Francis Reynolds, <i>Blocks and <math>F</math>-class algebras of finite groups</i> . . . . .	193
Richard Rochberg, <i>Which linear maps of the disk algebra are multiplicative</i> . . . . .	207
Gary Sampson, <i>Sharp estimates of convolution transforms in terms of decreasing functions</i> . . . . .	213
Stephen Scheinberg, <i>Fatou's lemma in normed linear spaces</i> . . . . .	233
Ken Shaw, <i>Whittaker constants for entire functions of several complex variables</i> . . . . .	239
James DeWitt Stein, <i>Two uniform boundedness theorems</i> . . . . .	251
Li Pi Su, <i>Homomorphisms of near-rings of continuous functions</i> . . . . .	261
Stephen Willard, <i>Functionally compact spaces, <math>C</math>-compact spaces and mappings of minimal Hausdorff spaces</i> . . . . .	267
James Patrick Williams, <i>On the range of a derivation</i> . . . . .	273