Pacific Journal of Mathematics

A FAMILY OF COUNTABLE HOMOGENEOUS GRAPHS

C. Ward Henson

Vol. 38, No. 1 March 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 1, 1971

A FAMILY OF COUNTABLE HOMOGENEOUS GRAPHS

C. WARD HENSON

Let $\mathscr{K}$ be the class of all countable graphs and let $\mathscr{K}_p$ be the class of all members of $\mathscr{K}$ which have no complete subgraphs of cardinality p. R. Rado has constructed a graph U which is universal for $\mathscr{K}$. In this paper U is shown to be homogeneous, in the sense of Fraissé. Also a simple construction is given of a graph G_p which is homogeneous and universal for $\mathscr{K}_p$ (for each $p \geqq 3$) and the structure of these graphs is investigated.

It is shown that if H is an infinite member of $\mathscr{K}_p$ then H can be embedded in G_p in such a way that every automorphism of H extends uniquely to an automorphism of G_p. A similar result holds for U. Also, U and G_3 have single-orbit automorphisms, while if $p > 3$, then G_p has no such automorphism. Finally, a result concerning vertex colorings of the graphs G_p is proved and used to give a new proof of a Theorem of Folkman on vertex colorings of finite graphs.

1. A graph G is a relational structure which consists of a nonempty set $|G|$ of vertices and an irreflexive, symmetric binary relation $R(G)$ on $|G|$. If $A \subset |G|$ is nonempty, let $G|A$ denote the induced subgraph of G which has vertex set A. Write $H \subset G$ to mean that H equals $G|A$ for some $A \subset |G|$. An embedding of H into G is an isomorphism of H onto an induced subgraph of G. If such an embedding exists we say that G *admits* H. If G and H are isomorphic we write $G \cong H$.

The complement graph of G is denoted by $\bar{G}$. K_p denotes a complete graph with p vertices (p an integer $\geqq 1$.) For each $v \in |G|$, G^v denotes the induced subgraph of G which has vertex set

$$\{w \mid (w, v) \in R(G)\} .$$

(The *valence* subgraph determined by v.) The induced subgraph of G obtained by removing a vertex v will be designated by $G - v$. The cardinality of the vertex set $|G|$ will be denoted by $c(G)$. Z denotes the set of all the integers and N the set of nonnegative integers.

The study of homogeneous relational structures was begun by Fraissé [4] as an attempt to generalize certain familiar properties of the ordering of the rational numbers. This study was continued in a very general setting by Jónsson [6 and 7] and by Morley and

Vaught [8]. The basic properties of homogeneous graphs needed in this paper may be summarized as follows.

DEFINITION 1.1. A graph G is *homogeneous* if whenever $H \subset G$ and $c(H) < c(G)$, every embedding of H into G can be extended to an automorphism of G.

THEOREM 1.2. *An infinite graph G is homogeneous $\longleftrightarrow$ whenever $H \subset G$, $c(H) < c(G)$ and $v \in |H|$, every embedding of $H - v$ into G can be extended to an embedding of H into G.*

THEOREM 1.3. *Let G be an infinite homogeneous graph.*

(a) *Suppose $c(H) = c(G)$ and G admits every graph $K \subset H$ for which $c(K) < c(H)$. Then G admits H.*

(b) *If H is homogeneous, $c(H) = c(G)$ and G and H admit exactly the same graphs of cardinality $< c(G)$, then $H \cong G$.*

In case G is a countably infinite graph, as will be true in this paper, Definition 1.1 comes from [4]; in that case, Theorem 1.2 is [4, Theorem 5.5] and Theorem 1.3 is [4, Theorems V and 5.4]. In general, G is homogeneous in the sense of Definition 1.1 if and only if G is $\mathscr{K}$-homogeneous in the sense of [7] and [8], where $\mathscr{K}$ is the class of all graphs; here Theorems 1.2 and 1.3 are included in [8, Theorems 2.3 and 2.5]. (It should be noted that in [8], and in model theory generally, "homogeneous" is used in a different, weaker sense. This should cause no confusion here, since only the meaning which agrees with [4] will be used.)

Rado's graph [9, 10] is universal among countable graphs by virtue of satisfying the condition

(A) if F_1, F_2 are disjoint, finite sets of vertices of G, then there is another vertex which is connected in G to every member of F_1 and to no member of F_2.

THEOREM 1.4. *Any graph G (with $c(G) = \aleph_0$) which satisfies condition (A) is homogeneous. Moreover, any two such graphs are isomorphic.*

Proof. Rado [10] showed that any graph which satisfies (A) must admit every finite graph. Thus the second statement follows from the first by Theorem 1.3.b.

Let G be a graph which satisfies (A) and $c(G) = \aleph_0$. We prove that G is homogeneous by showing that it satisfies the condition in Theorem 1.2. Suppose $H \subset G$ and $c(H) < c(G)$, so that H is finite.

Let $v \in |H|$ and assume f is an embedding of $H - v$ into G. Let $F_1 = f(|H^v|)$ and $F_2 = \text{Range}(f) - F_1$. There is a vertex w in $|G|$ which is connected to every member of F_1 and to no member of F_2. It follows that letting $f(v) = w$ extends f to an embedding of H into G, completing the proof.

We will designate by U a graph (isomorphic to Rado's graph) which is constructed as follows. Let $\{P_n \mid n \in N\}$ be an enumeration of the finite subsets of N, each one occurring infinitely often. Choose a sequence $v_0 < v_1 < \cdots$ in N which satisfies $v_n > \max(P_n)$ for all $n \in N$. To define U let $|U| = N$ and let $R(U)$ consist of all pairs of vertices of the form (w, v_n) or (v_n, w) where $w \in P_n$ and $n \in N$. Then U satisfies the following strong form of (*A*).

(A′) if $F \subset |U|$ is finite, then there exist arbitrarily large v in $|U|$ which satisfy

$$F = \{w \mid w < v \quad \text{and} \quad (w, v) \in R(U)\} .$$

In particular, U satisfies (A) and is thus isomorphic to Rado's graph, by Theorem 1.4. (Note that Rado's graph itself does not satisfy (A′).)

REMARK. In [2] Erdös and Renyi put a natural probability measure on the set of all graphs with vertex set N, and show that the measure of the set of such graphs which satisfy condition (A) is 1. They conclude from this that almost all graphs with vertex set N have a nontrivial automorphism. In fact the stronger result, that almost all such graphs are isomorphic to U, follows from Theorem 1.4.

COROLLARY 1.5. (a) $\bar{U} \cong U$

(b) *if* $|U| = A_1 \cup \cdots \cup A_n$ *and* $A_1, \cdots, A_n$ *are pairwise disjoint, then* $U|A \cong U$ *for some* $j = 1, \cdots, n$.

Proof. (a) $\bar{U}$ obviously satisfies condition (A).

(b) It suffices to consider the case $n = 2$.
Suppose $|U| = A \cup A'$ and $A \cap A' = \emptyset$, and assume that neither $U|A$ nor $U|A'$ is isomorphic to U. Then there exist disjoint, finite subsets F_1, F_2 of A and F_1', F_2' of A' which satisfy: (i) if v is connected in U to every member of F_1 and to no member of F_2, then $v \in A$, and (ii) if v is connected in U to every member of F_1' and to no member of F_2', then $v \in A'$. But $F_1 \cup F_1'$ and $F_2 \cup F_2'$ are disjoint, so there is a vertex v which is connected in U to every member of $F_1 \cup F_1'$ and to no member of $F_2 \cup F_2'$. This implies that $v \notin A \cup A'$, which is a contradiction.

It follows immediately from Theorem 1.5 that if $A \subset |U|$ and

$|U| - A$ is finite, then $U | A \cong U$. Also, using 1.5.a and the vertex symmetry of U we note that $U^v \cong (\bar{U})^v$, for any $v \in |U|$. Then since $|U^v|$ and $|(\bar{U})^v|$ form a partition of $|U - v|$ it follows that $U^v \cong U$ for every $v \in |U|$.

Recall that two graphs H_1, H_2 with the same vertex set are called *edge disjoint* if $R(H_1) \cap R(H_2) = \varnothing$. If $\mathscr{F}$ is a family of graphs with a common vertex set A, then the *union* of $\mathscr{F}$ is the graph whose vertex set is A and whose edge relation is $\bigcup \{R(H) \mid H \in \mathscr{F}\}$. *A spanning subgraph* of G is a graph H which satisfies $|H| = |G|$ and $R(H) \subset R(G)$.

THEOREM 1.6. *There is a family $\{H_i \mid i \in N\}$ of pairwise edge disjoint graphs* (all with vertex set N) *such that if $|G| = N$, $R(H_i) \subset R(G)$ and $R(H_j) \cap R(G) = \varnothing$* (for some $i, j \in N$) *then $G \cong U$.*

Proof. Let $\{(P_n, Q_n, f(n), g(n)) \mid n \in N\}$ be an enumeration of all quadruples (A, B, i, j) in which A, B are disjoint, finite subsets of N and $i, j \in N$. Let $v_0 < v_1 < \cdots$ be a sequence in N such that $v_n > \max (P_n \cup Q_n)$ for all $n \in N$. Define H_i, for each $i \in N$, by letting $|H_i| = N$ and letting $R(H_i)$ consist of all pairs of vertices (w, v_n) and (v_n, w) such that $f(n) = i$ and $w \in P_n$ or $g(n) = i$ and $w \in Q_n$.

Suppose $|G| = N$ and, for some $i, j \in N$, G satisfies $R(H_i) \subset R(G)$ and $R(H_j) \cap R(G) = \varnothing$. Let F_1, F_2 be disjoint, finite subsets of $|G|$. Choose n so that $P_n = F_1$, $Q_n = F_2$, $f(n) = i$ and $g(n) = j$. Then v_n is connected in H_i (and thus in G) to every member of F_1. Also v_n is connected in H_j (and thus not in G) to every member of F_2. This shows that G satisfies condition (A) and therefore G is isomorphic to U.

In particular, Theorem 1.6 asserts that the union of the family $\{H_i \mid i > 0\}$ is isomorphic to U. Thus there exists a family $\{G_i \mid i \in N\}$ of pairwise edge disjoint spanning subgraphs of U which satisfies (i) the union of the family is U, and (ii) if G is any spanning subgraph of U such that $R(G_i) \subset R(G)$, for some $i \in N$, then $G \cong U$.

Recall that a (one-way) Hamiltonian path for a graph G (with $c(G) = \aleph_0$) is a bijection τ from N onto $|G|$ such that for each n, $\tau(n)$ and $\tau(n+1)$ are connected in G. The path τ will be called *totally symmetric* if the function sending $\tau(n)$ to $\tau(n+1)$ (each $n \in N$) is an embedding of G into itself.

THEOREM 1.7. *There exists a totally symmetric, one-way Hamiltonian path for U.*

Proof. Let $\{P_n \mid n \in N\}$ be an enumeration of all finite subsets of N, with the properties:

(i) $P_n \subset \{0, \cdots, n\}$ for each $n \in N$, and (ii) each finite subset of N occurs in the list $\{P_n \mid n \in N\}$ infinitely often. For $n \in N$ define

$$a_n = 2 + \frac{n(n+1)}{2}$$

so that $a_0 = 2$ and $a_{n+1} = a_n + n + 1$. Construct a chain $Q_0 \subset Q_1 \subset \cdots$ of finite subsets of $N - \{0\}$ by letting $Q_0 = \{1\}$ and (for $n \geqq 0$)

$$Q_{n+1} = Q_n \bigcup \{a_{n+1} - k \mid k \in P_n\} .$$

If $k \in P_n$ then $0 \leqq k \leqq n$ so that

$$a_n + 1 = a_{n+1} - n \leqq a_{n+1} - k \leqq a_{n+1} .$$

It follows, by induction on n, that $Q_n \subset \{0, \cdots, a_n\}$ and

$$Q_{n+1} - Q_n \subset \{a_n + 1, \cdots, a_{n+1}\} .$$

Now let $A = \bigcup \{Q_n \mid n \in N\}$ and construct a graph G with $|G| = N$ and $R(G) = \{(m, n) \mid |m - n| \in A\}$. Since $1 \in A$, it is obvious that G has a totally symmetric (one-way) Hamiltonian path. Thus it sufficies to prove that G satisfies condition (A), so that $U \cong G$.

If F_1, F_2 are disjoint, finite subsets of N, we may choose n large enough so that $P_n = F_1$ and $F_1 \bigcup F_2 \subset \{0, \cdots, n\}$. For each $0 \leqq k \leqq n$ the construction of Q_{n+1} insures that

$$a_{n+1} - k \in Q_{n+1} \longleftrightarrow k \in F_1 .$$

But since $A \bigcap \{0, \cdots, a_{n+1}\} = Q_{n+1}$, it follows that

$$a_{n+1} - k \in A \longleftrightarrow k \in F_1 .$$

Thus a_{n+1} is connected in G to every member of F_1 and to no member of F_2. That is, G satisfies condition (A) and the proof is complete.

REMARK. Let Z be the set of all the integers and A the set constructed in the proof of Theorem 1.7. Define a graph H with $|H| = Z$ by letting

$$R(H) = \{(a, b) \mid a, b \in Z \text{ and } |a - b| \in A\} .$$

Evidently the functions f, sending a to $a + 1$, and g, sending a to $-a$, are automorphisms of H. Moreover, since $1 \in A$, the identity function from Z to $|H|$ defines a two-way Hamiltonian path for H. Finally, if F_1, F_2 are disjoint, finite subsets of $|H|$, choose k large enough so that $f^k(F_1 \bigcup F_2) \subset N$, and let $b \in N$ be connected in H to every member of $f^k(F_1)$ and to no member of $f^k(F_2)$. (Choose b using the fact that $H \mid N \cong U$, as proved above.) Then $f^{-k}(b)$ is connected in

H to every vertex in F_1 and to no vertex in F_2. That is, H satisfies condition (A) and is thus isomorphic to U.

This may be summarized by stating that U has a totally symmetric, two-way Hamiltonian path. In particular, note that U has an automorphism with a single orbit.

2. This section is devoted to a family $\{G_p \mid p \geqq 3\}$ of induced subgraphs of U, defined by letting

$$|G_p| = \{m \mid m \in N \text{ and there is no finite set } A \subset N \text{ with } m = \max A \text{ and } U \mid A \cong K_p\} ,$$

for each integer $p \geqq 3$. It follows that $G_p \subset G_{p+1} \subset U\ (p \geqq 3)$. and that U is the union of the chain of graphs $\{G_p \mid p \geqq 3\}$. In addition, G_p satisfies the following condition, analogous to (A).

(A_p) (i) G does not admit K_p,

(ii) if F_1, F_2 are disjoint, finite sets of vertices of G and $G \mid F_1$ does not admit K_{p-1}, then there is another vertex which is connected in G to every member of F_1 and to no member of F_2.

LEMMA 2.1. *For each $p \geqq 3$, G_p satisfies condition (A_p).*

Proof. It is obvious that G_p satisfies (i). Suppose F_1, F_2 are disjoint, finite subsets of $|G_p|$ and that $G_p \mid F_1$ does not admit K_{p-}. Since U satisfies (A') we may choose $v \in |U|$ which satisfies $v > \max (F_1 \cup F_2)$ and

$$F_1 = \{w \mid w < v \quad \text{and} \quad (w, v) \in R(U)\} .$$

It suffices to observe that $U \mid F_1 = G_p \mid F_1$ dose not admit K_{p-1} and therefore $v \in |G_p|$.

LEMMA 2.2. *Let $p \geqq 3$ and assume that G satisfies condition (A_p). Suppose also that H is a finite graph which does not admit K_p, $v \in |H|$ and f is an embedding of $H - v$ into G. Then f can be extended to an embedding of H into G.*

THEOREM 2.3. *For each $p \geqq 3$, G_p is homogeneous, and admits exactly those finite graphs which do not admit K_p. Moreover, any graph G (with $c(G) = \aleph_0$) which satisfies condition (A_p) is isomorphic to G_p.*

Proof. Using Lemma 2.2, it can be shown by induction on $c(H)$ that if G satisfies (A_p) and H is a finite graph which does not admit K_p, then G admits H. That is, any graph which satisfies (A_p) admits exactly those finite graphs which do not admit K_p.

It follows by Theorem 1.2 that if $c(G) = \aleph_0$ and G satisfies (A_p) then G is homogeneous. (In particular, by Lemma 2.1, G_p is homogeneous.) Finally, by Theorem 1.3.b, any such G is isomorphic to G_p.

The following result is an immediate consequence of Theorem 1.3.a and Theorem 2.3, and answers a question raised (for $p = 3$) by Erdös and Hajnal [3, p. 121].

COROLLARY 2.4. *For each* $p \geqq 3$, G_p *is a universal graph in the class of countable graphs which do not admit* K_p.

COROLLARY 2.5. *Let* $p \geqq 3$.
(a) *If* $A \subset |G_p|$ *and* $|G_n| - A$ *is finite, then* $G_p | A \cong G_p$
(b) *If* $v \in |G_{p+1}|$ *then* $(G_{p+1})^v \cong G_p$.

Proof. (a) If F_1, F_2 are disjoint, finite subsets of A and $G_p | F_1$ does not admit K_{p-1}, then there are, in fact, infinitely many vertices in $|G_p|$ which are connected to every member of F_1 and to no member of F_2. Since $|G_p| - A$ is finite, this shows that $G_p | A$ satisfies (A_p).

(b) Suppose H is a finite graph satisfying $H \subset (G_{p+1})^v$ and suppose that f is an embedding of H into $(G_{p+1})^v$. Since G_{p+1} is homogeneous, there is an automorphism g of G_{p+1} such that g extends f and $g(v) = v$. Thus g determines an automorphism of $(G_{p+1})^v$ which extends f. This shows that $(G_{p+1})^v$ is homogeneous. The fact that $(G_{p+1})^v$ and G_p are isomorphic follows from Theorems 1.3.b and 2.3 and the observation that $(G_{p+1})^v$ admits a finite graph H if and only if G_{p+1} admits the graph obtained from H by adding a new vertex connected to every member of $|H|$.

Note that for each $v \in |G_3|$ the graph $(G_3)^v$ is infinite, with no two vertices connected.

The analogue of Corollary 1.5.b for G_p is false, as can be seen by considering the partition of $|G_p|$ determined by $|(G_p)^v|$ and its complement. (Also see § 4.)

If H is a spanning subgraph of G_p $(p \geqq 3)$ and $H \neq G_p$, then H cannot be isomorphic to G_p. For there must be vertices a, b in $|G_p|$ which are connected in G_p but not in H. If $H \cong G_p$ then there exists $A \subset |G_p|$ so that $H | A \cup \{a\}$ and $H | A \cup \{b\}$ are isomorphic to K_{p-1}. But this would imply that $G_p | A \cup \{a, b\} \cong K_p$, which is impossible.

In particular, the analogue for G_p of Theorem 1.6 is false.

Corresponding to Theorem 1.7 are the following two results.

THEOREM 2.6 *There exists a totally symmetric (one-way) Hamiltonian path for* G_3.

Proof. Let the sequence $\{P_n \mid n \in N\}$ be as in the proof of Theorem 1.7, and construct a chain $Q_0 \subset Q_1 \subset \cdots$ of finite subsets of N-$\{0\}$ as follows. Let $Q_0 = \{1\}$; for $n \geqq 0$, if there exist $a, b \in P_n$ so that $0 < |a - b| \in Q_n$, then let $Q_{n+1} = Q_n$. Otherwise let

$$Q_{n+1} = Q_n \bigcup \{3^{n+1} - k \mid k \in P_n\} .$$

Recalling that $P_n \subset \{0, \cdots, n\}$, it follows that $Q_n \subset \{0, \cdots, 3^n\}$ and $Q_{n+1} - Q_n \subset \{3^n + 1, \cdots, 3^{n+1}\}$. Let $A = \bigcup \{Q_n \mid n \in N\}$ and construct a graph G, as in the proof of Theorem 1.7, by letting $|G| = N$ and

$$R(G) = \{(m, n) \mid |m - n| \in A\} .$$

As before, it suffices to prove that this graph satisfies condition (A_3).

Suppose that F_1, F_2 are disjoint, finite subsets of $|G|$ and that $G \mid F_1$ does not admit K_2. That is, if $a, b \in F_1$ and $a \neq b$ then $|a - b| \notin A$. Choose n large enough so that $P_n = F_1$ and

$$F_1 \cup F_2 \subset \{0, \cdots, n\} .$$

Since $Q_n \subset A$ there do not exist $a, b \in P_n$ with $0 < |a - b| \in Q_n$. Thus if $0 \leqq k \leqq n$ then $3^{n+1} - k \in Q_{n+1} \leftrightarrow k \in P_n$. It follows that 3^{n+1} is connected in G to every member of F_1 and to no member of F_2.

Suppose next that G admits K_3. It follows that there exist $0 < a < b$ such that $G \mid \{0, a, b\} \cong K_3$. That is, a, b and $b - a$ are in A. Let n be the smallest integer for which $a \in Q_n$. If $b \in Q_n$ then $n \geqq 1$, and a, $b \in Q_n - Q_{n-1}$ (since $a < b$.) But then $a = 3^n - c$ and $b = 3^n - d$, for some $c, d \in P_{n-1}$. Moreover $c - d = b - a \in A$ and $0 \leqq d < c \leqq n - 1$ so that $|c - d| \in Q_{n-1}$, contradicting the definition of Q_n. Therefore $b \notin Q_n$, and there exists $k \geqq n$ such that $b \in Q_{k+1} - Q_k$. If $b - a \in Q_{k+1} - Q_k$ we obtain a contradiction as above, by considering c, $d \in P_k$ with $b = 3^{k+1} - c$ and $b - a = 3^{k+1} - d$.

Since $b - a < b \in Q_{k+1}$ and $b - a \in A$, it follows that $b - a$ must be in Q_k. Thus a and $b - a$ are both $\leqq 3^k$ and therefore

$$b \leqq 2 \cdot 3^k < 3^{k+1} - k .$$

But since $b \in Q_{k+1} - Q_k$, which implies that $3^{k+1} - k \leqq b \leqq 3^{k+1}$, this is a contradiction. That is, G does not admit K_3.

This shows that G satisfies the condition (A_3) and therefore G is isomorphic to G_3, completing the proof.

As in the Remark following Theorem 1.7, it can be shown that G_3 has a totally symmetric, two-way Hamiltonian path. In particular, G_3 has an automorphism with a single orbit. In contrast, for the graphs G_p with $p \geqq 4$ we have the following result.

THEOREM 2.7. *If* $p \geqq 4$, *then there is no automorphism of* G_p *with a single orbit.*

Proof. If otherwise, we can construct a graph G with an automorphism f such that $G \cong G_p$, $|G| = Z$ and $f(a) = a + 1$ for all $a \in Z$. We let

$$A = \{a \mid (a, 0) \in R(G)\} .$$

It then follows that

$$R(G) = \{(a, b) \mid |a - b| \in A\} .$$

Since G_p admits K_{p-2}, there exist $a_1 < \cdots < a_{p-2}$ in $|G|$ so that $G \mid \{a_1, \cdots, a_{p-2}\} \cong K_{p-2}$. That is, if $1 \leqq i < j \leqq p-2$ then $a_j - a_i \in A$. Since G satisfies condition (A_p) there exists $a \in |G|$ which is connected in G to 0 but is not connected to any of the vertices $a_i - a_j$ (where $i \neq j$) and is distinct from them.

If a_i is connected in G to $a_j + a$, so that $|a_j + a - a_i|$ is in A, it follows that a is connected to $a_i - a_j$. Thus $i = j$. (Conversely, $|a| \in A$, so that a_i is connected to $a_i + a$.) If we let

$$\mathrm{B} = \{a_1, \cdots, a_{p-2}, a_1 + a, \cdots, a_{p-2} + a\} ,$$

it follows that $G \mid B$ admits K_{p-2} but not K_{p-1} (recall that $p \geqq 4$). Thus there exists a vertex k which is connected in G to every member of B.

Consider $C = \{0, a, k - a_1, \cdots, k - a_{p-2}\}$. If $i \neq j$ then

$$|(k - a_i) - (k - a_j)| = |a_i - a_j| \in A ,$$

so that

$$G \mid \{k - a_1, \cdots, k - a_{p-2}\} \cong K_{p-2} .$$

By the choice of k, $|k - a_i| \in A$ and $|k - a_i - a| \in A$. Thus each $k - a_i$ is connected in G to 0 and to a. Since a is connected to 0 in G by choice, it follows that $G \mid C \cong K_p$. This contradicts the fact that $G \cong G_p$, and completes the proof.

REMARK. It is easy to show that if G is a homogeneous graph, then so is $\bar{G}$. Thus the graphs $\bar{G}_p$ are all homogeneous, and evidently distinct from the graphs U and $G_p (p \geqq 3.)$ If G is a homogeneous graph, but not connected, the components of G must be complete (consider the induced subgraphs with two vertices which are not connected) and pairwise isomorphic (since G is vertex symmetric.) It is an interesting and apparently open question if there are any homogeneous graphs G (with $c(G) = \aleph_0$) which have G and $\bar{G}$ connected, other than U, G_p and $\bar{G}_p$ ($p \geqq 3$.)

The existence of the graphs G_p may be approached indirectly, by noting that the class $\mathscr{K}_p$ of all graphs which do not admit K_p satisfies the amalgamation property of [7] (property D in [4].) Thus, in the language of [7], G_p is the $\mathscr{K}_p$-homogeneous universal structure of cardinality $\aleph_0$.

3. This section is concerned with the problem of embedding an infinite graph H in U (or in one of the graphs G_p) in such a way that automorphisms of H extend to automorphisms of U (G_p.) In addition it is shown that each of these graphs has a maximal independent set M whose permutations all extend uniquely to automorphisms.

THEOREM 3.1. *Let H be a graph with $c(H) = \aleph_0$. There exists an embedding of H onto an induced subgraph $H' \subset U$ such that each automorphim of H' extends uniquely to an automorphism of U.*

Proof. Let $n_1 < n_2 < \cdots$ be an increasing sequence of positive integers. Construct a chain of graphs $H_0 \subset H_1 \subset H_2 \subset \cdots$ by letting $H_0 = H$ and continuing as follows. For $k \geqq 1$ obtain $|H_k|$ by adding to $|H_{k-1}|$ a new vertex $v(A, k)$ for each finite set $A \subset |H_{k-1}|$ such that $A \cap |H_0|$ has exactly n_k elements. Each new vertex $v(A, k)$ is connected in H_k to the vertices in A and to no others. (Recall that $H_{k-1} \subset H_k$ is also required.) Define K to be the union of the chain $\{H_k \mid k \geqq 0\}$ so that $H_k \subset K$ for each $k \geqq 0$ and, in particular, $H \subset K$.

If F_1, F_2 are disjoint, finite subsets of $|K|$, choose k large enough so that $F_1 \cup F_2 \subset |H_{k-1}|$ and $F_1 \cap |H_0|$ has at most n_k elements. Since $|H_0|$ is infinite there is a set $B \subset |H_0|$ such that $B \cap F_2 = \varnothing$, $F_1 \cap |H_0| \subset B$ and B has exactly n_k elements. Letting $A = F_1 \cup B$, it follows that $v(A, k)$ is a vertex in H_k which is connected in H_k (and thus in K) to every vertex in F_1 and to no vertex in F_2. This shows that K satisfies condition (A). Since only countably many vertices are added at each stage of the construction of K, it follows that $K \cong U$.

Any automorphism f of H_{k-1} which satisfies $f(|H_0|) = |H_0|$ can be extended to an automorphism of H_k by setting $f(v(A, k) =$

$v(f(A), k)$ (for each new vertex.) Moreover, since $f(v(A, k))$ must be connected in H_k to the vertices in $f(A)$ and no others, this is the only possible way to extend such an f. Therefore, each automorphism of H_0 can be extended to an automorphism of K, and this extension is unique among automorphism of K which leave each set $|H_k|$ invariant $(k > 0.)$

But the members of $|H_k|$ are distinguished, among vertices of K, by virtue of being in $|H_0|$ or being connected in K to at most n_k elements of $|H_0|$. Thus any automorphism of K which leaves $|H_0|$ invariant must also leave $|H_k|$ invariant, for each $k > 0$. That is, each automorphism of $H(=H_0)$ has a unique extension to an automorphism of $K \cong U$, completing the proof.

COROLLARY 3.2. *There is a maximal independent set of vertices $M \subset |U|$ such that every permutation of M extends uniquely to an automorphism of U.*

Proof. Let H be a graph with $\aleph_0$ vertices, no two connected, and carry out the construction in the proof of Theorem 3.1. Set $M = |H'| \subset |U|$ and note that every permutation of the set M is an automorphism of H', and thus extends uniquely to an automorphism of U. Since $n_k > 0$ (for $k \geqq 1$) each vertex in $|K| - |H|$ is connected to at least one member of $|H|$ in K. It follows that M is a maximal independent set of vertices in $|U|$ as desired.

To extend Theorem 3.1 to the homogeneous graphs G_p requires a modification of the construction given above. Fix $p \geqq 3$ and let H be any graph, with $c(H) = \aleph_0$, which does not admit K_p. Construct a chain $\{H_k \mid k \geqq 0\}$ by letting $H_0 = H$ and proceeding as above, except that $v(A, k)$ is a vertex in $|H_k| - |H_{k-1}|$ only when $A \cap |H_0|$ has n_k elements and $H_{k-1} | A$ does not admit K_{p-1}. (A any finite subset of $|H_{k-1}|$, $k \geqq 1$.) Letting K be the union of the chain $\{H_k\}$, it is easy to see that the restriction on adding new vertices at each stage insures that K does not admit K_p. Moreover, the same argument as above shows that each automorphism of $H(= H_0)$ extends uniquely to an automorphism of K.

It is not always true, however, that K satisfies condition (A_p). This difficulty can be overcome if we assume that H satisfies

(B) if $F_1 \subset |H|$ is finite, then there exists an infinite independent set $A \subset |H| - F_1$ such that no vertex in F_1 is connected in H to any vertex in A.

Assume now that H satisfies (B) and let F_1, F_2 be disjoint, finite subsets of $|K|$ such that $K | F_1$ does not admit K_{p-1}. Choose k large enough so that $F_1 \cup F_2 \subset |H_{k-1}|$ and $F_1 \cap |H_0|$ has at most n_k elements.

Let $F_3 \subset |H_0|$ consist of $F_1 \cap |H_0|$ together with every vertex in $|H_0|$ which is connected to some member of $F_1 - |H_0|$. Since F_1 is finite and each vertex in $|K| - |H_0|$ is connected to only finitely many members of $|H_0|$, it follows that F_3 is a finite set. Applying condition (B), there exists an infinite independent set A' in H_0 such that $A' \cap F_3 = \varnothing$ and no vertex in F_3 is connected in H_0 to any vertex in A'. In particular, $K | F_1 \cup A'$ does not admit K_{p-1}. Since A' is infinite, we may choose a set $B \subset (F_1 \cup A') \cap |H_0|$ such that $B \cap F_2 = \varnothing$, $F_1 \cap |H_0| \subset B$ and B has exactly n_k elements. Letting $A = F_1 \cup B$, it follows that $K | A$ does not admit K_{p-1} and $A \cap |H_0| = B$ has n_k elements. Thus $v(A, k)$ is a vertex in K which is connected to every member of F_1 and to no member of F_2. That is, K satisfies condition (A_p) whenever H satisfies condition (B).

THEOREM 3.3. *Let $p \geqq 3$ and suppose H is a graph with $c(H) = \aleph_0$ which does not admit K_p. Then there is an embedding of H onto an induced subgraph $H' \subset G_p$ such that each automorphism of H' extends uniquely to an automorphism of G_p.*

Proof. If H satisfies (B) then the proof has been given above. Otherwise, extend H to a graph H'' by adding a vertex v'' for each $v \in |H|$, connecting v'' only to v in H''. Then $H \subset H''$ and H'' clearly does not admit K_p. If F_1 is a finite subset of $|H''|$ then letting $A = \{v'' \mid v \in |H| - F_1\} - F_1$ shows that H'' satisfies condition (B). Finally, note that each automorphism f of H extends uniquely to an automorphism of H'' (by setting $f(v'') = (f(v)''$.) The desired embedding of H into G_p is thus obtained by restricting to H an appropriate embedding of H'' into G_p.

COROLLARY 3.4. *For each $p \geqq 3$ there exists a maximal independent set of vertices $M \subset |G_p|$ such that every permutation of M extends uniquely to an automorphism of G_p.*

Proof. Proceed as in the proof of Corollary 3.2, noting that the graph H with $\aleph_0$ vertices, no two connected, satisfies condition (B).

THEOREM 3.5. *Let G be U or G_p for some $p \geqq 3$ and let*

$$a_1, \cdots, a_n \in |G| .$$

There is an automorphism f of G which has $a_1, \cdots, a_n$ as its only fixed points.

Proof. Let H' be $G | \{a_1, \cdots, a_n\}$. Obtain H from H' by adding a set $C = \{v_n \mid n \in Z\}$ of new vertices, but without adding any new edges. Obviously H can be embedded in G and H satisfies (B). Let $c(H') < n_1 < n_2 < \cdots$ and using the sequence $\{n_k\}$ carry out the appropriate construction (as in the proof of Theorem 3.1 or Theorem 3.3.) We obtain a graph K which is isomorphic to G and satisfies $H \subset K$. Moreover, K has an automorphism f which satisfies $f(v) = v$ (if v is one of $a_1, \cdots, a_n$) and $f(v_n) = v_{n+1}$ (if $n \in Z$). If $v = v(A, k)$ is any member of $|K| - |H|$, suppose $f(v) = v$. It follows that $f(A) = A$, and hence that $f(A \cap |H|) = A \cap |H|$. Now $A \cap |H|$ has $n_k > c(H')$ elements, so that $A \cap C \neq \emptyset$. Moreover, $f(A \cap C) = A \cap C$, which implies that $A \supset C$, contradicting the fact that A is a finite set. Thus f has no fixed points in $|K| - |H|$ and therefore has only $a_1, \cdots, a_n$ as fixed points. Finally note that there is an isomorphism g of K onto G so that $g(v) = v$ if $v \in \{a_1, \cdots, a_n\}$. The automorphism $g \circ f \circ g^{-1}$ of G has as its fixed points only $a_1, \cdots, a_n$, and is therefore the desired function.

4. It is well known that there are finite graphs of arbitrarily large chromatic number which do not admit K_3 (eg. [1].) Thus for each $p \geqq 3$ the graph G_p has chromatic number $\aleph_0$. This may be expressed by saying that if $|G_p| = A_1 \cup \cdots \cup A_n$ then for some $j = 1, \cdots, n$ $G_p \mid A_j$ admits K_2. The results of this section amount to a strengthening of this fact.

THEOREM 4.1. *Let* $p \geqq 3$ *and suppose* $|G_p| = A_1 \cup A_2$. *Then either there exists* $B \subset A_1$ *such that* $A_1 - B$ *is finite and* $G_p \mid B \cong G_p$ *or* $G_p \mid A_2$ *admits every finite graph which does not admit* K_p.

Proof. Let A_1, A_2 be as above for G_p and suppose that the desired set B does not exist. Construct a sequence $\{(C_n, D_n) \mid n \geqq 1\}$, where C_n, D_n are disjoint, finite subsets of A_1 (for each $n \geqq 1$) as follows. Since $G_p \mid A_1$ is not isomorphic to G_p, it fails to satisfy condition (A_p). Thus there exist disjoint, finite subsets (C_1, D_1) of A_1 such that $G_p \mid C_1$ does not admit K_{p-1} and every vertex in $|G_p|$ which is connected to every member of C_1 and to no member of D_1 lies in A_2.

Assuming that $(C_1, D_1), \cdots, (C_n, D_n)$ have been constructed, let $E_n = \bigcup \{C_j \cup D_j \mid j = 1, \cdots, n\}$ so that E_n is a finite subset of A_1. Since $G_p \mid A_1 - E_n$ is not isomorphic to G_p there exist disjoint, finite subsets (C_{n+1}, D_{n+1}) of $A_1 - E_n$ such that $G_p \mid C_{n+1}$ does not admit K_{p-1} and every vertex in $|G_p|$ which is connected to every member of C_{n+1} and to no member of D_{n+1} lies in $A_2 \cup E_n$.

Now let H be any finite graph which does not admit K_p and suppose $|H| = \{a_1, \cdots, a_n\}$. For convenience assume that $|H| \cap |G_p| = \emptyset$. Construct a graph G with vertex set $|G| = |H| \cup E_n$ so that $G|(|H|) = H$, $G|E_n = G_p|E_n$ and each a_j in $|H|$ is connected in G to every element of C_j and to no element of $E_n - C_j$. If $G|F \cong K_p$, then $F \cap |H| \neq \emptyset$ and $F \cap E_n \neq \emptyset$. Since each vertex in E_n is connected in G to at most one member of $|H|$ it follows that

$$F \cap |H| = \{a_j\} \text{ (for some } j = 1, \cdots, n) \quad \text{and} \quad F \cap E_n \subset C_j .$$

That is, $G|C_j$ $(= G_p|C_j)$ admits K_{p-1}, which is a contradiction. Therefore G does not admit K_p.

Since G_p is homogeneous, there is an embedding f of G into G_p such that $f(v) = v$ for each $v \in E_n$. Therefore $f(a_j) \notin E_n$(for each $j = 1, \cdots, n$) and $f(a_j)$ is connected in G_p to every vertex in C_j and to no vertex in D_j. By the construction of (C_j, D_j) it follows that $f(a_j) \in A_2$. That is, f maps H into $G_p|A_2$, showing that $G_p|A_2$ admits every finite graph which does not admit K_p.

COROLLARY 4.2. *Let $p \geqq 3$ and suppose that $|G_p| = A_1 \cup \cdots \cup A_n$. Then for some $j = 1, \cdots, n$ the graph $G_p|A_j$ admits every finite graph which does not admit K_p.*

Proof. By induction on n, using Theorem 4.1.

We raise the question of whether or not the conclusion of Corollary 4.2 can be strengthened to read: "$G_p|A_j$ admits G_p, for some $j = 1, \cdots, n$." ?

COROLLARY 4.2 is equivalent to the following result of Folkman [5] concerning finite graphs, which he proved by entirely different methods.

COROLLARY 4.3. (Folkman) *Let $p \geqq 3$, $n \geqq 2$ and suppose G is any finite graph which does not admit K_p. There exists a finite graph H, which also does not admit K_p, such that if $|H| = A_1 \cup \cdots \cup A_n$, then for some $j = 1, \cdots, n$, $H|A_j$ admits G.*

The proof of this equivalence is a standard application of (for example) König's Infinity Lemma, as in the proof of the Erdös-de Bruijn Theorem which states that an infinite graph G has chromatic number $\geqq k$ if and only if it has a finite induced subgraph with chromatic number $\geqq k (k \in N)$. Thus the details will be omitted.

F. Galvin has raised the question of whether or not an "edge coloring" version of Corollary 4.3 holds when $p = 3$. (See [3] for a discussion of this and related problems.) It seems possible that further investigation of G_3 might shed some light on this problem.

The author is indebted to Fred Galvin for his useful comments on an earlier version of this paper.

REFERENCES

1. Blanche Descartes, *Solution to advanced problem* 4526, Amer. Math. Monthly, **61** (1954), 352.
2. P. Erdös, and A. Renyi, *Asymmetric graphs*, Acta Math. Acad. Sci. Hungar. **14** (1963), 295-315.
3. P. Erdös and A. Hajnal, *Problems and results in finite and infinite combinatorial analysis*, Annals of the New York Academy of Sciences (International Conference on Combinatorial Mathematics) **175**, article 1 (1970), 115-124.
4. Roland Fraiseé, *Sur l'extension aux relations de quelques proprietés des ordres*, Ann. Sci. École Norm. Sup. (3) **71** (1954), 361-388.
5. J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math., **18** (1970), 19-24.
6. B. Jónsson, *Universal relational systems*, Math. Scand. **4** (1956), 193-208.
7. ———, *Homogeneous universal relational systems*, Math. Scand. **8** (1960), 137-142.
8. M. Morley, and R. Vaught, *Homogeneous universal models*, Math. Scand., **11** (1962), 37-57.
9. R. Rado, *Universal graphs and universal functions*, Acta Arith., **9** (1964), 331-340.
10. ———, *Universal graphs, in F. Harary*, ed., *A Seminar in Graph Theory*, Holt, Rinehart and Winston (New York, 1967), 83-85.

Received February 17, 1971.

DUKE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 38, No. 1 March, 1971