A SUFFICIENT CONDITION FOR $L^p$-MULTIPLIERS

SATORU IGARI AND SHIGEHIKO KURATSUBO
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Suppose $1 \leq p \leq \infty$. For a bounded measurable function $\phi$ on the $n$-dimensional euclidean space $R^n$ define a transformation $T_\phi$ by $(T_\phi f)^* = \hat{\phi} \hat{f}$, where $f \in L^2 \cap L^p(R^n)$ and $\hat{f}$ is the Fourier transform of $f$:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(x)e^{-i\xi x} \, dx.$$  

If $T_\phi$ is a bounded transform of $L^p(R^n)$ to $L^p(R^n)$, $\phi$ is said to be $L^p$-multiplier and the norm of $\phi$ is defined as the operator norm of $T_\phi$.

**Theorem 1.** Let $2n/(n + 1) < p < 2n/(n - 1)$ and $\phi$ be a radial function on $R^n$, so that, it does not depend on the arguments and may be denoted by $\phi(r)$, $0 \leq r < \infty$. If $\phi(r)$ is absolutely continuous and

$$M = \|\phi\|_{\infty} + \left( \sup_{R > 0} R \int_{R}^{2R} \frac{d}{dr} \phi(r) \right)^{1/2},$$

then $\phi$ is an $L^p$-multiplier and its norm is dominated by a constant multiple of $M$.

To prove this theorem we introduce the following notations and Theorem 2. For a complex number $\delta = \sigma + i\tau$, $\sigma > -1$, and a reasonable function $f$ on $R^n$ the Riesz-Bochner mean of order $\delta$ is defined by

$$s^\delta_n(f, x) = \frac{1}{(2\pi)^{n/2}} \int_{|\xi| < R} \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta \hat{f}(\xi)e^{i\tau \xi} d\xi.$$  

Put

$$t^\delta_n(f, x) = s^\delta_n(f, x) - s^{\delta - 1}_n(f, x)$$

and define the Littlewood-Paley function by

$$g^\delta_n(f, x) = \left( \int_0^\infty \frac{|t^\delta_n(f, x)|^2}{R} \, dR \right)^{1/2},$$

which is introduced by E. M. Stein in [3]. Then we have the following.

**Theorem 2.** If $2n/(n + 2\sigma - 1) < p < 2n/(n - 2\sigma + 1)$ and $1/2 < \sigma < (n + 1)/2$, then

$$A \|g^\delta_n(f)\|_p \leq \|f\|_p < A' \|g^\delta_n(f)\|_p,$$

where $A$ and $A'$ are positive constants depending only on $n$, $\sigma$, and $p$. (Note that $g^\delta_n$ is a bounded operator from $L^p$ to itself.)
where $A$ and $A'$ are constants not depending on $f$.

The first part of inequalities is proved by E. M. Stein [3] for $p = 2$ and by G. Sunouchi [4] for $2n/(n + 2\sigma - 1) < p < 2$. The other parts will be shown by the conjugacy method as in S. Igari [2], so that we shall give a sketch of a proof.

**Proof of Theorem 2.** For $\hat{\delta} = \sigma + i\tau, \sigma > -1$, and $t > 0$ let $K^t(\delta)$ be the Fourier transform of $[\max \{(1 - |\xi|^{2})^{-t}, 0\}]^\delta$. Since $K^t(\delta)$ is radial, we denote it simply by $K^t(r), r = |x|$. Then $K^t(r) = 2^t I^t(\hat{\delta} + 1) V_{(n/2)+t}(rt)t^s$, where $V_s(s) = J_s(s)s^{-\delta}$ and $J_s$ denotes the Bessel function of the first kind. Considering the Fourier transform of $t^s_k(f, x)$ we get

$$t^s_k(f, x) = \frac{1}{\sqrt{2\pi^3}} \int_{\mathbb{R}^n} f(y) T^s_k(x - y) dy = f \ast T^s_k(x),$$

where $T^s_k(x) = R^{-2} A K^{-s-1}(x)$ and $A = \partial^2/\partial x^2 + \cdots + \partial^2/\partial x^2$.

Let $H$ be the Hilbert space of functions on $(0, \infty)$ whose inner product is defined by $\langle f, g \rangle = \int_0^\infty f_R \overline{g}_R R^{-4} dR$. For a function $g_R(x)$ in $L^2(\mathbb{R}^n; H)$, that is, $H$-valued $L^2$-function, define an operator $v^s$ by

$$v^s(g, x) = \frac{1}{\sqrt{2\pi^3}} \int_{\mathbb{R}^n} \langle T^s_k(y), \overline{g}_R(x - y) \rangle dy.$$

By the associativity of convolution relation

$$\int_{\mathbb{R}^n} v^s(g, x) \overline{f}(x) dx = \int_{\mathbb{R}^n} \langle g(x), t^s(f, x) \rangle dx$$

for every $f$ in $L^2(\mathbb{R}^n)$ and $g$ in $L^2(\mathbb{R}^n; H)$, which implies that $v^s$ is the adjoint of $t^s$.

By the Plancherel formula

$$\| t^s(f) \|_{L^2(H)} = \left( \int_{\mathbb{R}^n} \left( 1 - \frac{|\xi|^2}{R^2} \right)^{2\sigma - 2} \frac{|\xi|^4}{R^2} dR \right)^{1/2} \| f \|_{L^2}$$

$$= B_{\sigma} \| f \|_{L^2},$$

where $B_{\sigma} = [B(2\sigma - 1, 2)]^{1/2}, \hat{\delta} = \sigma + i\tau$, and $\sigma > 1/2$. Therefore $f = (1/B\hat{\delta}) v^s t^s(f)$ for $f \in L^2(\mathbb{R}^n)$. By Schwarz inequality $|\langle t^s(f, x), g(x) \rangle| \leq \| t^s(f, x) \|_H \| g(x) \|_H$. Applying this inequality with (2) to (1) we get

$$\| v^s(g) \|_{L^2} \leq B_{\sigma} \| g \|_{L^2(H)}.$$

On the other hand
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\[ \int_{|x| > 2|y|} \| T_\delta(x + y) - T_\delta(x) \|_H dx < A e^{-|x|/2} \]

for $\sigma > \alpha + 1$, $\alpha = (n - 1)/2$ (see [4]), where $A_{p,q}$ denotes here and after a constant depending only on $p, q$ and the dimension $n$. Thus by the well-known argument (see, for example, Dunford-Schwartz [1; p. 1171] we get

(4) \[ \| v^t(f) \|_{L^q(H)} \leq A_{p,q} e^{-|x|/2} \| f \|_{L^q} \]

and

(5) \[ \| v^s(g) \|_{L^p} \leq A_{p,q} e^{-|x|/2} \| g \|_{L^p(H)} \]

for $1 < q \leq 2$ and $\delta = \rho + i\sigma$, $\rho > \alpha + 1$. Fix such a $\rho$ and a $q$.

Remark that the Stein's interpolation theorem (see [5; p. 100]) is valid for $H$-valued $L^p$-spaces and apply it between the inequalities (2) and (4), and (3) and (5). Then we get

(6) \[ \| v^s(f) \|_{L^p(H)} \leq A_{p,q} \| f \|_{L^p} \]

and

(7) \[ \| v^t(g) \|_{L^p} \leq A_{p,q} \| g \|_{L^p(H)} \]

for $1 < p \leq 2$ and $\sigma > (n/p) - \alpha$.

Since $f = (1/B^s) v^t f$, we get Theorem 2 for $2n/(n + 2\sigma - 1) < p \leq 2$ from (6) and (7). The case where $2 \leq p < 2n/(n - 2\sigma + 1)$ is proved by the equality (1) and the conjugacy method.

**Proof of Theorem 1.** Let $f \in L^2(\mathbb{R}^n)$. By definition

(8) \[ t_\delta(T_\delta f, x) = -\frac{1}{\sqrt{2\pi^n}} \int_{|\xi| < n} \frac{\xi^2}{R^2} \phi(\xi) \hat{f}(\xi) e^{ix\xi} d\xi . \]

Put

\[ F(r\omega) = \frac{-1}{\sqrt{2\pi^n}} \int_{|\xi| = n} \frac{\xi^2}{R^2} \hat{f}(\xi) e^{ix\xi} , \]

where $r = |\xi|$ and $\omega$ is a unit vector. Then

\[ t_\delta(T_\delta f, x) = \int_0^R \phi(r) \left( \int_{|\omega| = 1} F(r\omega) d\omega \right) r^{n-1} dr . \]

The last term is, by integration by parts, equal to

\[ \phi(R) \int_0^R r^{n-1} dr \int_{|\omega| = 1} F(r\omega) d\omega - \int_0^R \frac{d}{dr} \phi(r) dr \int_0^r s^{n-1} ds \int_{|\omega| = 1} F(s\omega) d\omega . \]

Thus
\[ t_n(T \phi f, x) = \phi(R) t_n(f, x) - \int_0^R \frac{d}{dr} \phi(r) \frac{r^2}{R^2} t_n(f, x) dr. \]

By the Schwarz inequality the last integral is, in absolute value, dominated by
\[
\left( \frac{1}{R} \int_0^R \left| \frac{d}{dr} \phi(r) \right|^2 r^2 dr \right)^{1/2} \left( \frac{1}{R^2} \int_0^R \left| t_n(f, x) \right|^2 r^2 dr \right)^{1/2}.\]

Divide \((0, R)\) into the intervals of the form \((R/2^{j+1}, R/2^j)\) and dominate \(r^2\) by \(R^2/2^j\) in each interval. Then the first integral is bounded by
\[
\sum_{j=0}^{\infty} \frac{1}{2^{2j}} \int_{R/2^{j+1}}^{R/2^j} \left| \frac{d}{dr} \phi(r) \right|^2 dr \leq 4 \sup_{R>0} R \int_0^{2R} \left| \frac{d}{dr} \phi(r) \right|^2 dr.
\]

Therefore
\[
g_n^*(T \phi f, x) \leq \| \phi \|_n \left( \int_0^\infty \left| t_n^*(f, x) \right|^2 dR \right)^{1/2} + 2 \left( \sup_{R>0} \int_0^{2R} \left| \frac{d}{dr} \phi(r) \right|^2 dr \right)^{1/2} \left( \int_0^\infty \left| t_n^*(f, x) \right|^2 r^2 dr \int_0^{\infty} dR \right)^{1/2} \leq \frac{2}{\sqrt{3}} M g_n^*(f, x). \]

Thus, if \(2n/(n + 1) < p < 2n/(n - 1)\), then by Theorem 2 we have
\[
\| T \phi f \|_p \leq A' \| g_n^*(T \phi (f)) \|_p \leq \frac{2}{\sqrt{3}} A'M \| g_n^*(f) \|_p \leq \frac{2}{\sqrt{3}} AA'M \| f \|_p,
\]
which completes the proof.

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References
