

# Pacific Journal of Mathematics

**A SUFFICIENT CONDITION FOR  $L^p$ -MULTIPLIERS**

SATORU IGARI AND SHIGEHICO KURATSUBO

# A SUFFICIENT CONDITION FOR $L^p$ -MULTIPLIERS

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Suppose  $1 \leq p \leq \infty$ . For a bounded measurable function  $\phi$  on the  $n$ -dimensional euclidean space  $R^n$  define a transformation  $T_\phi$  by  $(T_\phi f)^\wedge = \phi \hat{f}$ , where  $f \in L^2 \cap L^p(R^n)$  and  $\hat{f}$  is the Fourier transform of  $f$ :

$$\hat{f}(\xi) = \hat{f} \frac{1}{\sqrt{2\pi^n}} \int_{R^n} f(x) e^{-i\xi x} dx .$$

If  $T_\phi$  is a bounded transform of  $L^p(R^n)$  to  $L^p(R^n)$ ,  $\phi$  is said to be  $L^p$ -multiplier and the norm of  $\phi$  is defined as the operator norm of  $T_\phi$ .

**THEOREM 1.** Let  $2n/(n+1) < p < 2n/(n-1)$  and  $\phi$  be a radial function on  $R^n$ , so that, it does not depend on the arguments and may be denoted by  $\phi(r)$ ,  $0 \leq r < \infty$ . If  $\phi(r)$  is absolutely continuous and

$$M = \|\phi\|_\infty + \left( \sup_{R>0} R \int_R^{2R} \left| \frac{d}{dr} \phi(r) \right|^2 dr \right)^{1/2} < \infty ,$$

then  $\phi$  is an  $L^p$ -multiplier and its norm is dominated by a constant multiple of  $M$ .

To prove this theorem we introduce the following notations and Theorem 2. For a complex number  $\delta = \sigma + i\tau$ ,  $\sigma > -1$ , and a reasonable function  $f$  on  $R^n$  the Riesz-Bochner mean of order  $\delta$  is defined by

$$s_R^\delta(f, x) = \frac{1}{\sqrt{2\pi^n}} \int_{|\xi| < R} \left( 1 - \frac{|\xi|^2}{R^2} \right)^\delta \hat{f}(\xi) e^{i\xi x} d\xi .$$

Put

$$t_R^\delta(f, x) = s_R^\delta(f, x) - s_{R^{-1}}^{\delta-1}(f, x)$$

and define the Littlewood-Paley function by

$$g_\delta^*(f, x) = \left( \int_0^\infty \frac{|t_R^\delta(f, x)|^2}{R} dR \right)^{1/2} ,$$

which is introduced by E. M. Stein in [3]. Then we have the following.

**THEOREM 2.** If  $2n/(n+2\sigma-1) < p < 2n/(n-2\sigma+1)$  and  $1/2 < \sigma < (n+1)/2$ , then

$$A \|g_\sigma^*(f)\|_p \leq \|f\|_p < A' \|g_\sigma^*(f)\|_p ,$$

where  $A$  and  $A'$  are constants not depending on  $f$ .

The first part of inequalities is proved by E. M. Stein [3] for  $p = 2$  and by G. Sunouchi [4] for  $2n/(n + 2\sigma - 1) < p < 2$ . The other parts will be shown by the conjugacy method as in S. Igari [2], so that we shall give a sketch of a proof.

*Proof of Theorem 2.* For  $\delta = \sigma + i\tau$ ,  $\sigma > -1$ , and  $t > 0$  let  $K_t^\delta(x)$  be the Fourier transform of  $[\max\{(1 - |\xi|^2 t^{-2}), 0\}]^\delta$ . Since  $K_t^\delta(x)$  is radial, we denote it simply by  $K_t^\delta(r)$ ,  $r = |x|$ . Then  $K_t^\delta(r) = 2^\delta \Gamma(\delta + 1) V_{(n/2) + \delta}(rt) t^n$ , where  $V_\beta(s) = J_\beta(s) s^{-\beta}$  and  $J_\beta$  denotes the Bessel function of the first kind. Considering the Fourier transform of  $t_R^\delta(f, x)$  we get

$$t_R^\delta(f, x) = \frac{1}{\sqrt{2\pi^n}} \int_{\mathbf{R}^n} f(y) T_R^\delta(x - y) dy = f * T_R^\delta(x),$$

where  $T_R^\delta(x) = R^{-2} \Delta K_R^{\delta-1}(x)$  and  $\Delta = \partial^2/(\partial x_1^2) + \dots + \partial^2/(\partial x_n^2)$ .

Let  $H$  be the Hilbert space of functions on  $(0, \infty)$  whose inner product is defined by  $\langle f, g \rangle = \int_0^\infty f_R \bar{g}_R R^{-1} dR$ . For a function  $g_R(x)$  in  $L^1(\mathbf{R}^n; H)$ , that is,  $H$ -valued  $L^1(\mathbf{R}^n)$ -function, define an operator  $v^\delta$  by

$$v^\delta(g, x) = \frac{1}{\sqrt{2\pi^n}} \int_{\mathbf{R}^n} \langle T_R^\delta(y), \bar{g}_R(x - y) \rangle dy.$$

By the associativity of convolution relation

$$(1) \quad \int_{\mathbf{R}^n} v^\delta(g, x) \bar{f}(x) dx = \int_{\mathbf{R}^n} \langle g(x), \bar{t}^\delta(f, x) \rangle dx$$

for every  $f$  in  $L^2(\mathbf{R}^n)$  and  $g$  in  $L^2(\mathbf{R}^n; H)$ , which implies that  $v^\delta$  is the adjoint of  $t^\delta$ .

By the Plancherel formula

$$(2) \quad \begin{aligned} \|t^\delta(f)\|_{L^2(H)} &= \left( \int_{|\xi|}^\infty \left(1 - \frac{|\xi|^2}{R^2}\right)^{2\sigma-2} \frac{|\xi|^4}{R^5} dR \right)^{1/2} \|f\|_{L^2} \\ &= B_\sigma \|f\|_{L^2}, \end{aligned}$$

where  $B_\sigma = [B(2\sigma - 1, 2)/2]^{1/2}$ ,  $\delta = \sigma + i\tau$ , and  $\sigma > 1/2$ . Therefore  $f = (1/B_\sigma^2) v^\delta t^\delta(f)$  for  $f \in L^2(\mathbf{R}^n)$ . By Schwarz inequality  $|\langle t^\delta(f, x), g(x) \rangle| \leq \|t^\delta(f, x)\|_H \|g(x)\|_H$ . Applying this inequality with (2) to (1) we get

$$(3) \quad \|v^\delta(g)\|_{L^2} \leq B_\sigma \|g\|_{L^2(H)}.$$

On the other hand

$$\int_{|x|>2|y|} \|T_R^\delta(x+y) - T_R^\delta(x)\|_R dx < A_\sigma e^{\tau/2}$$

for  $\sigma > \alpha + 1$ ,  $\alpha = (n - 1)/2$  (see [4]), where  $A_{p,q}$  denotes here and after a constant depending only on  $p, q$  and the dimension  $n$ . Thus by the well-known argument (see, for example, Dunford-Schwartz [1; p. 1171] we get

$$(4) \quad \|t^\delta(f)\|_{L^q(H)} \leq A_{q,\rho} e^{\tau/2} \|f\|_{L^q}$$

and

$$(5) \quad \|v^\delta(g)\|_{L^q} \leq A_{q,\rho} e^{\tau/2} \|g\|_{L^q(H)}$$

for  $1 < q \leq 2$  and  $\delta = \rho + i\tau$ ,  $\rho > \alpha + 1$ . Fix such a  $\rho$  and a  $q$ .

Remark that the Stein's interpolation theorem (see [5; p. 100]) is valid for  $H$ -valued  $L^p$ -spaces and apply it between the inequalities (2) and (4), and (3) and (5). Then we get

$$(6) \quad \|t^\sigma(f)\|_{L^p(H)} \leq A_{p,\sigma} \|f\|_{L^p}$$

and

$$(7) \quad \|v^\sigma(g)\|_{L^p} \leq A_{p,\sigma} \|g\|_{L^p(H)}$$

for  $1 < p \leq 2$  and  $\sigma > (n/p) - \alpha$ .

Since  $f = (1/B_\sigma^2)v^\sigma t^\sigma(f)$ , we get Theorem 2 for  $2n/(n + 2\sigma - 1) < p \leq 2$  from (6) and (7). The case where  $2 \leq p < 2n/(n - 2\sigma + 1)$  is proved by the equality (1) and the conjugacy method.

*Proof of Theorem 1.* Let  $f \in L^2(\mathbf{R}^n)$ . By definition

$$(8) \quad t_R^1(T_\phi f, x) = -\frac{1}{\sqrt{2\pi^n}} \int_{|\xi|<R} \frac{|\xi|^2}{R^2} \phi(\xi) \hat{f}(\xi) e^{i\xi x} d\xi.$$

Put

$$F(r\omega) = F(\xi) = \frac{-1}{\sqrt{2\pi^n}} \frac{|\xi|^2}{R^2} \hat{f}(\xi) e^{i\xi x},$$

where  $r = |\xi|$  and  $\omega$  is a unit vector. Then

$$t_R^1(T_\phi f, x) = \int_0^R \phi(r) \left( \int_{|\omega|=1} F(r\omega) d\omega \right) r^{n-1} dr.$$

The last term is, by integration by parts, equal to

$$\phi(R) \int_0^R r^{n-1} dr \int_{|\omega|=1} F(r\omega) d\omega - \int_0^R \frac{d}{dr} \phi(r) dr \int_0^r s^{n-1} ds \int_{|\omega|=1} F(s\omega) d\omega.$$

Thus

$$t_r^*(T_\phi f, x) = \phi(R)t_R^*(f, x) - \int_0^R \frac{d}{dr} \phi(r) \frac{r^2}{R^2} t_r^*(f, x) dr .$$

By the Schwarz inequality the last integral is, in absolute value, dominated by

$$\left( \frac{1}{R} \int_0^R \left| \frac{d}{dr} \phi(r) \right|^2 r^2 dr \right)^{1/2} \left( \frac{1}{R^3} \int_0^R |t_r^*(f, x)|^2 r^2 dr \right)^{1/2} .$$

Divide  $(0, R)$  into the intervals of the form  $(R/2^{j+1}, R/2^j)$  and dominate  $r^2$  by  $R^2/2^{2j}$  in each interval. Then the first integral is bounded by

$$\sum_{j=0}^{\infty} \frac{1}{2^{j-1}} \frac{R}{2^{j+1}} \int_{R/2^{j+1}}^{R/2^j} \left| \frac{d}{dr} \phi(r) \right|^2 dr \leq 4 \sup_{R>0} R \int_R^{2R} \left| \frac{d}{dr} \phi(r) \right|^2 dr .$$

Therefore

$$\begin{aligned} g_1^*(T_\phi f, x) &\leq \|\phi\|_\infty \left( \int_0^\infty \frac{|t_r^*(f, x)|^2}{R} dR \right)^{1/2} \\ &\quad + 2 \left( \sup_{R>0} R \int_R^{2R} \left| \frac{d}{dr} \phi(r) \right|^2 dr \right)^{1/2} \left( \int_0^\infty |t_r^*(f, x)|^2 r^2 dr \int_r^\infty \frac{dR}{R^4} \right)^{1/2} \\ &\leq \frac{2}{\sqrt{3}} M g_1^*(f, x) . \end{aligned}$$

Thus, if  $2n/(n+1) < p < 2n/(n-1)$ , then by Theorem 2 we have

$$\|T_\phi f\|_p \leq A' \|g_1^*(T_\phi f)\|_p \leq \frac{2}{\sqrt{3}} A' M \|g_1^*(f)\|_p \leq \frac{2}{\sqrt{3}} AA' M \|f\|_p ,$$

which completes the proof.

Finally the authors wish to express their thanks to the referee by whom the proof Theorem 2 is simplified.

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Received October 12, 1970.

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