

Pacific Journal of Mathematics

**FIXED POINT THEOREMS FOR NONLINEAR NONEXPANSIVE
AND GENERALIZED CONTRACTION MAPPINGS**

WILLIAM A. KIRK

FIXED POINT THEOREMS FOR NONLINEAR NONEXPANSIVE AND GENERALIZED CONTRACTION MAPPINGS

W. A. KIRK

Let X be a reflexive Banach space, H a closed convex subset of X , and let K be a nonempty, bounded, closed and convex subset of H which possesses normal structure. If $T: K \rightarrow H$ is nonexpansive and if $T: \partial_H K \rightarrow K$ where $\partial_H K$ denotes the boundary of K relative to H , then T has a fixed point in K . This result generalizes an earlier theorem of the author, and a more recent theorem of F. E. Browder. An analogue is given for generalized contraction mappings in conjugate spaces.

1. Introduction. In [13] we proved that if K is a nonempty, bounded, closed and convex subset of a reflexive Banach space, and if K possesses "normal structure" (defined below), then every nonexpansive mapping T of K into K has a fixed point. This result, also proved independently by F. E. Browder [4] and D. Göhde [11] (in uniformly convex spaces), initiated rather extensive study of fixed point theory for nonlinear nonexpansive operators in Banach spaces, including applications to the study of nonlinear equations of evolution by Browder [5] and to certain nonlinear functional equations (see Browder and Petryshyn [8], Kolomý [16], Srinivasacharyulu [21]).

In this paper we modify the approach of [13] to treat the following problem: Given closed and convex subsets K and H of a Banach space X such that $K \cap H \neq \emptyset$ and an operator $T: K \rightarrow X$ such that (i) $T: K \cap H \rightarrow H$ and (ii) $T: \partial_H K \rightarrow K$ (where $\partial_H K$ denotes the relative boundary of $K \cap H$ in H), when does T have a fixed point? This kind of problem has been of particular interest in the case where the operator T is completely continuous, H is the positive cone of X , and the fixed points of T correspond to positive solutions of a differential equation (for example, see [17]). A standard approach is to use the technique of "radial projection" to associate with T an operator B which is also completely continuous, has the same fixed points as T , and maps the intersection of H with the ball $K: \|x\| \leq R$ into itself, thus permitting application of the classical Schauder Theorem [19]. Such an approach, however, is not suitable for our purposes because we consider mappings of nonexpansive type. Since radial projection is in general *not* nonexpansive (see [9]), the associated operator B need not be nonexpansive and one cannot obtain a fixed point by direct application of the theorem of [13].

Before stating our results we establish relevant notation and definitions.

A mapping T of a subset K of a Banach space X into X is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

For a subset S of a Banach space X , the symbol $\delta(S)$ denotes the *diameter* of S —i.e.,

$$\delta(S) = \sup \{ \|x - y\|; x, y \in S \}.$$

The notation $U(z; r)$ is used to denote the spherical neighborhood of z of radius $r > 0$:

$$U(z; r) = \{x \in X: \|z - x\| < r\}.$$

Similarly,

$$\bar{U}(z; r) = \{x \in X: \|z - x\| \leq r\}.$$

The concept of normal structure, due to Brodskii and Milman [3], plays a key role in our approach. A bounded convex set K in a Banach space X is said to have *normal structure* if for each convex subset S of K which contains more than one point, there is a point $x \in S$ which is a *nondiametral point* of S (i.e., $\sup \{ \|x - y\|; y \in S \} < \delta(S)$). *Compact convex sets possess normal structure* ([3], [10, Lemma 1]) *as do all bounded convex subsets of uniformly convex spaces.* (For a comparison of normal structure and uniform convexity, see Belluce-Kirk-Steiner [2]. The concept has also been studied by Gossez and Lami Dozo [12].)

We wish to thank the referee for his suggestions, particularly for pointing out the corollary to Theorem 3.1.

2. A fixed point theorem for nonexpansive mappings. For H and K subsets of X , we use the symbol $\partial_H K$ to denote the boundary of K relative to H : Thus, letting $H - K$ denote the points of H which are not in K , if K is closed,

$$\partial_H K = \{z \in K: U(z; r) \cap (H - K) \neq \emptyset \text{ for each } r > 0\}.$$

THEOREM 2.1. *Let X be a reflexive Banach space, H a closed convex set in X , and K a nonempty, bounded, closed, convex subset of H which possess normal structure. If $T: K \rightarrow H$ is nonexpansive, and if $T: \partial_H K \rightarrow K$, then T has a fixed point in K .*

The above theorem immediately reduces to our theorem of [13] upon taking $H = K$. A more interesting consequence of this theorem arises from taking $H = X$:

COROLLARY. *Let K be a bounded closed convex subset of a reflexive Banach space X and suppose K possesses normal structure. If $T: K \rightarrow X$ is nonexpansive, and if T maps the boundary of K into K , then T has a fixed point in K .*

Browder first obtained the above result [6, Theorem 3] for K a bounded closed convex set in a uniformly convex space with the additional assumption that T is defined on an open convex set $G \supset K$ with $\text{dist}(K, X - G) > 0$. Subsequently Browder [7] and Nussbaum [18] have removed this assumption (in a uniformly convex setting) while proving more general results, a fact which is significant because in general one may not enlarge the domain of non-expansive mapping [20].

Proof of Theorem 2.1. Let \mathcal{S} be the family of all closed convex subsets of H such that for $F \in \mathcal{S}$, $F \cap K \neq \emptyset$ and $T: F \cap K \rightarrow F$. Since $H \in \mathcal{S}$, $\mathcal{S} \neq \emptyset$. Let $\{F_\alpha\}$ be a descending chain of sets of \mathcal{S} , and let $F = \bigcap_\alpha F_\alpha$. Note that $F \cap K$ is nonempty, since each of the sets $F_\alpha \cap K$ is a nonempty weakly compact subset of X . Also, since $T: F_\alpha \cap K \rightarrow F_\alpha$ for each α , clearly $T: F \cap K \rightarrow F$. Since F is closed and convex, $F \in \mathcal{S}$, and therefore by Zorn's Lemma, \mathcal{S} has a minimal element.

Letting F be such a minimal element of \mathcal{S} , first note that we may assume $\partial_r K \neq \emptyset$, for otherwise $F \subset K$ and $T: F \cap K \rightarrow F$ would imply $T: F \rightarrow F$. The existence of a fixed point would then follow from the theorem of [13].

Now we assume $\delta(F \cap K) > 0$ and obtain a contradiction. Let $\delta = \delta(F \cap K)$. Since K possesses normal structure, there exists a point $c \in F \cap K$ such that

$$\sup \{ \|c - z\| : z \in F \cap K \} = r < \delta.$$

Let

$$C = \{x \in X : F \cap K \subset \bar{U}(x; r)\}.$$

It is easily seen that C is closed and convex and, since $c \in F \cap C$, $(F \cap C) \cap K \neq \emptyset$. Also there exist points $x, y \in F \cap K$ such that $\|x - y\| > r$. Such points cannot be elements of C and therefore $F \cap C$ is a proper subset of F . We complete the proof by showing $F \cap C \in \mathcal{S}$. Since we have already seen that $(F \cap C) \cap K \neq \emptyset$, we need only show that $T: (F \cap C) \cap K \rightarrow F \cap C$.

Suppose $z \in (F \cap C) \cap K$. Let

$$W = \bar{U}(Tz; r) \cap F.$$

If $W \in \mathcal{S}$, then since $W \subset F$ and F is minimal, $W = F$. This implies

$$F \cap K \subset F \subset \bar{U}(Tz; r),$$

and hence $Tz \in C$. Since $T: F \cap K \rightarrow F$, this in turn yields $Tz \in F \cap C$. Therefore $T: (F \cap C) \cap K \rightarrow F \cap C$ if $W \in \mathcal{S}$ for every $z \in (F \cap C) \cap K$. We complete the proof by showing this to be the case.

First suppose $x \in W \cap K$. Then $x \in F \cap K$ so $\|x - z\| \leq r$ (because $z \in C$). Hence $\|Tx - Tz\| \leq r$ and $Tx \in \bar{U}(Tz; r)$. But $x \in W \cap K$ also implies $x \in F \cap K$ and hence $Tx \in F$. Therefore $Tx \in \bar{U}(Tz; r) \cap F = W$, i.e., $T: W \cap K \rightarrow W$.

Finally, since $\partial_F K \neq \emptyset$, it follows that $W \cap K \neq \emptyset$. To see this, note that if $y \in \partial_F K$ then $y \in F \cap K$ and $\|y - z\| \leq r$, which implies $\|Ty - Tz\| \leq r$ and therefore $Ty \in W$. But also $\partial_F K \subset \partial_H K$ implies $Ty \in K$; hence $Ty \in W \cap K$ and $W \cap K \neq \emptyset$.

This completes the proof that $F \cap C \in \mathcal{S}$, contradicting the assumption $\delta(F \cap C) > 0$. Thus $\delta(F \cap C) = 0$ and $F \cap C$ consists of a single point which, because $T: \partial_F K \rightarrow K$, is fixed under T .

3. Generalized contraction mappings. In this section we give an analogue of Theorem 2.1 for the class of generalized contraction mappings studied in [14, 15]. With X a Banach space, and $K \subset X$, a mapping $T: K \rightarrow X$ is called a *generalized contraction mapping* if for each $x \in K$ there is a number $\alpha(x) < 1$ such that

$$\|Tx - Ty\| \leq \alpha(x)\|x - y\| \quad \text{for each } y \in K.$$

It was noted in Belluce-Kirk [1] that mappings of this type provide an example of a class of mappings with "diminishing orbital diameters"; thus fixed point theorems proved in [1] apply to this class of mappings. In [15] it is shown that if A is a bounded open convex subset of X and if $F: A \rightarrow X$ is continuously Fréchet differentiable on A , then F is a generalized contraction mapping on A if and only if for each $x_0 \in A$ the norm of the Fréchet derivative F'_{x_0} of F at x_0 is less than one. It is also shown that if K is a w^* -compact convex subset of a conjugate Banach space X and if $T: K \rightarrow K$ is a generalized contraction mapping, then T has a fixed point in K . This result may be generalized as follows:

THEOREM 3.1. *Let X be a conjugate Banach space, H a convex w^* -closed subset of X , and K a nonempty convex w^* -compact subset of H . If $T: K \rightarrow H$ is a generalized contraction mapping on K , and if $T: \partial_H K \rightarrow K$, then T has a fixed point in K .*

Proof. As in the proof of Theorem 2.1, obtain a w^* -compact convex set F minimal with respect to the properties $F \cap K \neq \emptyset$ and

$T: F \cap K \rightarrow F$. As before, it may be assumed that $\partial_x(K) \neq \emptyset$ (otherwise $F \subset K$ and existence of a fixed point follows from Theorem 1.1 of [15]).

The argument parallels that of Theorem 2.1 upon obtaining a point $c \in F \cap K$ such that

$$(1) \quad \sup \{ \|c - z\| : z \in F \cap K \} < \delta .$$

Such a point can be obtained by letting $x \in \partial_F K$, noting that $Tx \in F \cap K$, and using the procedure of the proof of Theorem 1.1 of [15] to show that Tx has the property specified for c in (1). Specifically, one can show that if $\delta = \delta(F \cap K) > 0$ then for the number $\alpha(x) < 1$ associated with T by definition, one has

$$\bar{U}(Tx; \alpha(x)\delta) \cap F \in \mathcal{F}$$

which implies

$$\sup \{ \|Tx - z\| : z \in F \cap K \} \leq \alpha(x)\delta = r < \delta .$$

Then letting $Tx = c$, define the set C as in Theorem 2.1 and observe that

$$C = \bigcap_{x \in F \cap K} \bar{U}(x; r) .$$

Thus C is w^* -compact and convex. This and the fact that the set W defined later in the argument is also w^* -compact and convex, enables one to complete the proof precisely as in Theorem 2.1. We omit the details.

COROLLARY. *If X is a conjugate Banach space and H is a closed convex subset of X of which every intersection with a w^* -compact set is w^* -compact (e.g. $H = X$), and if $T: H \rightarrow H$ is a generalized contraction mapping on H , then T has a fixed point in H .*

Proof. Let $x \in H$ and let

$$K = H \cap \bar{U}\left(x; \frac{\|x - Tx\|}{1 - \alpha(x)}\right) .$$

Then $T: \partial_H K \rightarrow K$ and by Theorem 3.1, T has a fixed point in H .

REFERENCES

1. L. P. Belluce and W. A. Kirk, *Fixed point theorems for certain classes of non-expansive mappings*, Proc. Amer. Math. Soc., **20** (1969), 141-146.
2. L. P. Belluce, W. A. Kirk, and E. F. Steiner, *Normal structure in Banach spaces*, Pacific J. Math., **26** (1968), 433-440.

3. M. S. Brodskii and D. P. Milman, *On the center of a convex set*, Dokl. Acad. Nauk SSSR **59**(1948), 837-840.
4. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci., **54**(1965), 1041-1044.
5. ———, *Periodic solutions of nonlinear equations of evolution in infinite dimensional spaces*, Lecture Series in Differential Equations, University of Maryland, 1966 (AFOSR 65-1886).
6. ———, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc., **73**(1967), 875-881.
7. ———, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc., **74**(1968), 660-665.
8. F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., **72**(1966), 571-575.
9. D. G. De Figueiredo and L. A. Karlovitz, *On the radial projection in normed spaces*, Bull. Amer. Math. Soc., **73**(1967), 364-368.
10. R. De Marr, *Common fixed points for commuting contraction mappings*, Pacific J. Math., **13**(1963), 1139-1141.
11. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr., **30**(1965), 251-258.
12. J. P. Gossez and E. Lami Dozo, *Structure normale et base de Schauder*, Bull. de l'Acad. royale de Belgique (5^e Serie) **55**(1969), 673-681.
13. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72**(1965), 1004-1006.
14. ———, *On nonlinear mappings of strongly semicontractive type*, J. Math. Anal. Appl., **27**(1969), 409-412.
15. ———, *Mappings of generalized contractive type*, J. Math. Anal. Appl., **32**(1970), 567-572.
16. J. Kolomý, *Some existence theorems for nonlinear problems*, Comment. Math. Univ. Carolinae **7**(1966), 207-212.
17. M. A. Krasnosel'skii, *Positive solutions of operator equations*, Groningen, P. Noordhoff Ltd., 1964.
18. R. D. Nussbaum, *The fixed point index and asymptotic fixed point theorems for k -set-contractions*, Bull. Amer. Math. Soc., **75**(1969), 490-495.
19. J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Mathematica, **2**(1930), 171-180.
20. O. Schönbeck, *On the extension of Lipschitz maps*, Ark. Mat., **7**(1967), 201-209.
21. K. Srinivasacharyulu, *On some non-linear problems*, Canad. J. Math. **20**(1968), 394-397.

Received March 20, 1970 and in revised form September 2, 1970. Research supported by National Science Foundation grants GP-8367 and GP-18045.

THE UNIVERSITY OF IOWA.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
NAVAL WEAPONS CENTER

Pacific Journal of Mathematics

Vol. 38, No. 1

March, 1971

Bruce Alan Barnes, <i>Banach algebras which are ideals in a Banach algebra</i>	1
David W. Boyd, <i>Inequalities for positive integral operators</i>	9
Lawrence Gerald Brown, <i>Note on the open mapping theorem</i>	25
Stephen Daniel Comer, <i>Representations by algebras of sections over Boolean spaces</i>	29
John R. Edwards and Stanley G. Wayment, <i>On the nonequivalence of conservative Hausdorff methods and Hausdorff moment sequences</i>	39
P. D. T. A. Elliott, <i>On the limiting distribution of additive functions (mod 1)</i>	49
Mary Rodriguez Embry, <i>Classifying special operators by means of subsets associated with the numerical range</i>	61
Darald Joe Hartfiel, <i>Counterexamples to a conjecture of G. N. de Oliveira</i>	67
C. Ward Henson, <i>A family of countable homogeneous graphs</i>	69
Satoru Igari and Shigehiko Kuratsubo, <i>A sufficient condition for L^p-multipliers</i>	85
William A. Kirk, <i>Fixed point theorems for nonlinear nonexpansive and generalized contraction mappings</i>	89
Erwin Kleinfeld, <i>A generalization of commutative and associative rings</i>	95
D. B. Lahiri, <i>Some restricted partition functions. Congruences modulo 11</i>	103
T. Y. Lin, <i>Homological algebra of stable homotopy ring π_* of spheres</i>	117
Morris Marden, <i>A representation for the logarithmic derivative of a meromorphic function</i>	145
John Charles Nichols and James C. Smith, <i>Examples concerning sum properties for metric-dependent dimension functions</i>	151
Asit Baran Raha, <i>On completely Hausdorff-completion of a completely Hausdorff space</i>	161
M. Rajagopalan and Bertram Manuel Schreiber, <i>Ergodic automorphisms and affine transformations of locally compact groups</i>	167
N. V. Rao and Ashoke Kumar Roy, <i>Linear isometries of some function spaces</i>	177
William Francis Reynolds, <i>Blocks and F-class algebras of finite groups</i>	193
Richard Rochberg, <i>Which linear maps of the disk algebra are multiplicative</i>	207
Gary Sampson, <i>Sharp estimates of convolution transforms in terms of decreasing functions</i>	213
Stephen Scheinberg, <i>Fatou's lemma in normed linear spaces</i>	233
Ken Shaw, <i>Whittaker constants for entire functions of several complex variables</i>	239
James DeWitt Stein, <i>Two uniform boundedness theorems</i>	251
Li Pi Su, <i>Homomorphisms of near-rings of continuous functions</i>	261
Stephen Willard, <i>Functionally compact spaces, C-compact spaces and mappings of minimal Hausdorff spaces</i>	267
James Patrick Williams, <i>On the range of a derivation</i>	273