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**A REPRESENTATION FOR THE LOGARITHMIC DERIVATIVE
OF A MEROMORPHIC FUNCTION**

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A new representation is developed for the logarithmic derivative of a meromorphic function f in terms of its zeros and poles, using as parameters some of the critical points of f . Applications are made to locating all but a finite number of critical points of f .

1. The principal result.

THEOREM 1.1. *Let f be a meromorphic function of finite order ρ possessing the finite zeros a_1, a_2, a_3, \dots and poles b_1, b_2, b_3, \dots . Let $\zeta_1, \zeta_2, \dots, \zeta_n$, be any $n = [\rho]$ distinct zeros of the derivative f' of f which are not also zeros of f . Then for $z \neq a_j, b_j$ ($j = 1, 2, 3, \dots$)*

$$(1.1) \quad \frac{f'(z)}{f(z)} = \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(a_j)(z - a_j)} - \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(b_j)(z - b_j)}$$

where $\psi(z) = 1$ for $n = 0$,

$$(1.2) \quad \psi(z) = (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n) \quad \text{for } n > 0.$$

In (1.1) the convergence is uniform on every compact set excluding all the a_j and b_j .

In the case that f is a rational function with m zeros and p poles, identity (1.1) reduces to the familiar formula

$$f'(z)/f(z) = \sum_{j=1}^m (z - a_j)^{-1} - \sum_{j=1}^p (z - b_j)^{-1}.$$

Furthermore, if the second summation is omitted in (1.1), identity (1.1) reduces to one which we had previously obtained [See 1] for entire functions of finite order.

2. *Proof.* Being a meromorphic function, f can be written as a ratio of two entire functions, each of which has an Hadamard representation in terms of its zeros. Thus,

$$(2.1) \quad f(z) = z^m e^{P(z)} \prod_{j=1}^{\infty} [E(z/a_j, p)/E(z/b_j, q)]$$

where m is an integer (positive, negative or zero); $P(z)$ is a polynomial of degree at most $n = [\rho]$; p and q are nonnegative integers not exceeding n and

$$E(u, p) = (1 - u) \exp [u + (1/2)u^2 + \cdots + (1/p)u^p]$$

if $p > 0$ whereas $E(u, 0) = (1 - u)$. Taking the logarithmic derivative of (2.1) and simplifying, one finds that

$$(2.2) \quad \frac{f'(z)}{f(z)} = \frac{m}{z} + P'(z) + A(z) - B(z)$$

where

$$(2.3) \quad A(z) = \sum_{j=1}^{\infty} \frac{z^p}{a_j^p(z - a_j)}, \quad B(z) = \sum_{j=1}^{\infty} \frac{z^q}{b_j^q(z - b_j)}.$$

By hypothesis, $f'(\zeta_k) = 0$, $k = 1, 2, \dots, n$. Hence, from (2.2), follows that for $k = 1, 2, \dots, n$

$$(2.4) \quad P'(\zeta_k) = -(m/\zeta_k) - A(\zeta_k) + B(\zeta_k).$$

Since $P'(z)$ is a polynomial of degree at most $n - 1$, it can be represented by the Lagrange Interpolation Formula as

$$\frac{P'(z)}{\psi'(z)} = \sum_{k=1}^n \frac{P'(\zeta_k)}{\psi'(\zeta_k)(z - \zeta_k)}.$$

Hence, using (2.3) and (2.4), one finds that

$$(2.5) \quad \begin{aligned} \frac{P'(z)}{\psi'(z)} = & - \sum_{k=1}^n \frac{m}{\zeta_k \psi'(\zeta_k)(z - \zeta_k)} - \sum_{k=1}^n \sum_{j=1}^{\infty} \frac{\zeta_k^p}{a_j^p \psi'(\zeta_k)(z - \zeta_k)(\zeta_k - a_j)} \\ & + \sum_{k=1}^n \sum_{j=1}^{\infty} \frac{\zeta_k^q}{b_j^q \psi'(\zeta_k)(z - \zeta_k)(\zeta_k - b_j)}. \end{aligned}$$

In view of the fact that sums $A(z)$ and $B(z)$ are uniformly and absolutely convergent on every compact set that omits all the a_j and b_j , the order of summation of the double sums in (2.5) can be reversed. Thus the first double sum in (2.5) becomes

$$(2.6) \quad \sum_{j=1}^{\infty} \frac{1}{a_j^p} \sum_{k=1}^n \frac{\zeta_k^p}{\psi'(\zeta_k)(\zeta_k - a_j)(z - \zeta_k)} = \sum_{j=1}^{\infty} \frac{1}{a_j^p(z - a_j)} \left[\frac{S(z)}{\psi'(z)} - \frac{S(a_j)}{\psi'(a_j)} \right]$$

where

$$(2.7) \quad S(z) = \psi'(z) \sum_{k=1}^n \frac{\zeta_k^p}{(z - \zeta_k)\psi'(\zeta_k)}.$$

Since the polynomial $S(z)$ is of degree at most $n - 1$ with $S(\zeta_k) = \zeta_k^p$, the polynomial

$$(2.8) \quad T(z) = S(z) - z^p$$

is of degree at most n such that

$$T(\zeta_k) = 0, \quad \text{for } k = 1, 2, \dots, n.$$

Therefore $T(z) = c \psi(z)$, where c is a constant that may be zero. Accordingly,

$$S(z) = z^p + c \psi(z)$$

and the sum (2.6) becomes

$$(2.9) \quad \sum_{j=1}^{\infty} \frac{z^p}{\psi(z) a_j^p (z - a_j)} - \sum_{j=1}^{\infty} \frac{1}{\psi(a_j) (z - a_j)}.$$

Similarly the second double sum in (2.5) reduces to

$$(2.10) \quad \sum_{j=1}^{\infty} \frac{z^q}{\psi(z) b_j^q (z - b_j)} - \sum_{j=1}^{\infty} \frac{1}{\psi(b_j) (z - b_j)}.$$

Finally, on use of the Lagrange Interpolation Formula for $1/\psi(z)$, the single sum in (2.5) becomes

$$(2.11) \quad \frac{m}{z} \sum_{k=1}^n \left[\frac{1}{\zeta_k} + \frac{1}{z - \zeta_k} \right] \frac{1}{\psi'(\zeta_k)} = \frac{m}{z} \left[-\frac{1}{\psi(0)} + \frac{1}{\psi(z)} \right].$$

Substituting from (2.9), (2.10) and (2.11) into (2.5), one reduces (2.2) to

$$(2.12) \quad \frac{f'(z)}{f(z)} = \frac{m\psi(z)}{z\psi(0)} + \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(a_j)(z - a_j)} - \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(b_j)(z - b_j)}.$$

However, the first term here may be dropped since it is obtainable from the first or second sum in (2.12) by allowing either $m a_j$ (if $m > 0$) or $-m b_j$ (if $m < 0$) to coalesce at 0. Thus identity (1.1) has been established.

3. Location of critical points. An immediate consequence of Theorem 1.1 is the following:

THEOREM 3.1. *Let f be a meromorphic function of finite order ρ possessing the finite zeros a_1, a_2, a_3, \dots and poles b_1, b_2, b_3, \dots and let $\zeta_0, \zeta_1, \dots, \zeta_n$, be any $n + 1 = [\rho] + 1$ distinct critical points of f which are not also zeros of f . Then*

$$(3.1) \quad \sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - a_j)(\zeta_1 - a_j) \cdots (\zeta_n - a_j)} = \sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - b_j)(\zeta_1 - b_j) \cdots (\zeta_n - b_j)}.$$

Equation (3.1) follows from (1.1) on setting $z = \zeta_0$, writing out $\psi(a_j)$ according to (1.2) and cancelling the factor $\psi'(\zeta_0) \neq 0$.

As an application of (3.1), the following will now be proved.

THEOREM 3.2. *Let D_a and D_b be two regions with which can be associated a set R of points ζ such that a ray from ζ to some point γ separates \bar{D}_a from \bar{D}_b and such that inequality*

$$0 < \arg [(\gamma - \zeta)/(z - \zeta)] < \pi/(n + 1) \pmod{2\pi}$$

holds for all z in one of the regions D_a, D_b and inequality

$$-\pi/(n + 1) < \arg [\gamma - \zeta]/(z - \zeta) < 0 \pmod{2\pi}$$

holds for all z in the other region. Let f , a meromorphic function of finite order ρ , have all its zeros in D_a and all its poles in D_b . Then at most $n = [\rho]$ critical points of f lie in R .

Proof. If on the contrary $n+1$ distinct critical points $\zeta_0, \zeta_1, \dots, \zeta_n$ were in R , identity (3.1) holds for them in relation to the zeros and poles of f . By hypothesis, one can associate with each ζ_k , a point γ_k such that for $j = 1, 2, 3, \dots$ the inequalities

$$(3.2) \quad 0 < \arg \frac{\gamma_k - \zeta_k}{a_j - \zeta_k} < \frac{\pi}{n + 1} \pmod{2\pi}$$

$$(3.3) \quad -\frac{\pi}{n + 1} < \arg \frac{\gamma_k - \zeta_k}{b_j - \zeta_k} < 0 \pmod{2\pi}$$

hold (or those with a_j and b_j interchanged).

Setting

$$T(z) = \prod_{k=0}^n [(\gamma_k - \zeta_k)/(z - \zeta_k)] ,$$

one infers that

$$0 < \arg T(a_j) < \pi, \quad -\pi < \arg T(b_j) < 0 ,$$

for all j . This means that

$$0 < \arg \sum_{j=1}^{\infty} T(a_j) < \pi ,$$

$$-\pi < \arg \sum_{j=1}^{\infty} T(b_j) < 0 .$$

Consequently,

$$\sum_{j=1}^{\infty} T(a_j) \neq \sum_{j=1}^{\infty} T(b_j) ,$$

in contradiction to (3.1). Consequently, at most n distinct critical points ζ_k can lie on R , as was to be proved.

As an illustration, let f be a meromorphic function of order ρ ,

$1 \leq \rho < \infty$, and let

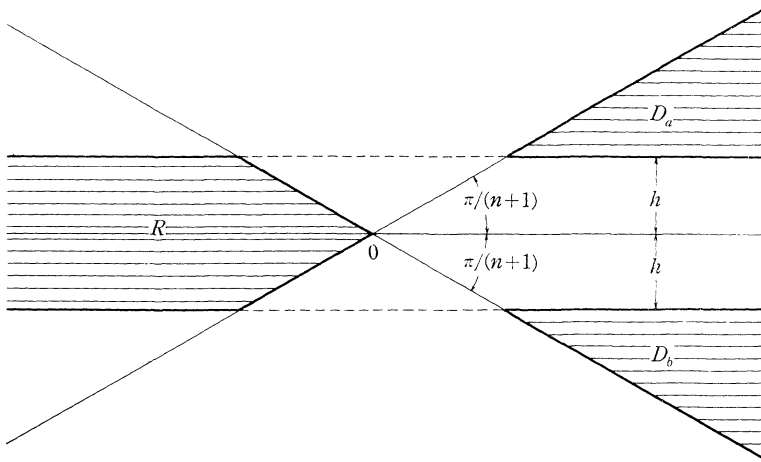


FIGURE 1

$$D_a = \{z = x + iy: x > 0, \quad 0 \leq h < y < x \tan [\pi/(n + 1)]\}$$

$$D_b = \{z = x + iy: x > 0, \quad -h > y > -x \tan [\pi/(n + 1)]\} .$$

Then, according to Theorem 3.2, at most n critical points of f lie in the region

$$R = \{z = x + iy: x < 0, \quad |y| < \min [h, |x| \tan [\pi/(n + 1)]]\}$$

REMARK. In identity (1.1) and the subsequent theorems, $f'(z)$ may be replaced by the linear combination $f'(z) + \lambda f(z)$ or more generally by

$$F_1(z) = f'(z) + f(z)g'(z)$$

where $g(z)$ is an arbitrary polynomial of degree at most n , provided $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_n$ are taken as the zeros of $F_1(z)$. This follows from the fact that the meromorphic function $F(z) = e^{g(z)}f(z)$ is also of order ρ , has the same zeros and poles as f and $F'(z) = e^{g(z)}F_1(z)$.

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