A REPRESENTATION FOR THE LOGARITHMIC DERIVATIVE
OF A MEROMORPHIC FUNCTION

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A new representation is developed for the logarithmic derivative of a meromorphic function \( f \) in terms of its zeros and poles, using as parameters some of the critical points of \( f \). Applications are made to locating all but a finite number of critical points of \( f \).

1. The principal result.

THEOREM 1.1. Let \( f \) be a meromorphic function of finite order \( \rho \) possessing the finite zeros \( a_1, a_2, a_3, \ldots \) and poles \( b_1, b_2, b_3, \ldots \). Let \( \zeta_1, \zeta_2, \ldots, \zeta_n \), be any \( n = [\rho] \) distinct zeros of the derivative \( f' \) of \( f \) which are not also zeros of \( f \). Then for \( z \neq a_j, b_j \) (\( j = 1, 2, 3, \ldots \))

\[
\frac{f'(z)}{f(z)} = \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(a_j)(z - a_j)} - \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(b_j)(z - b_j)}
\]

where \( \psi(z) = 1 \) for \( n = 0 \),

\[
\psi(z) = (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n) \quad \text{for } n > 0.
\]

In (1.1) the convergence is uniform on every compact set excluding all the \( a_j \) and \( b_j \).

In the case that \( f \) is a rational function with \( m \) zeros and \( p \) poles, identity (1.1) reduces to the familiar formula

\[
\frac{f'(z)}{f(z)} = \sum_{j=1}^{m} (z - a_j)^{-1} - \sum_{j=1}^{p} (z - b_j)^{-1}.
\]

Furthermore, if the second summation is omitted in (1.1), identity (1.1) reduces to one which we had previously obtained [See 1] for entire functions of finite order.

2. Proof. Being a meromorphic function, \( f \) can be written as a ratio of two entire functions, each of which has an Hadamard representation in terms of its zeros. Thus,

\[
f(z) = z^m e^{P(z)} \prod_{j=1}^{\infty} \left[ E(z/a_j, p)/E(z/b_j, q) \right]
\]

where \( m \) is an integer (positive, negative or zero); \( P(z) \) is a polynomial of degree at most \( n = [\rho] \); \( p \) and \( q \) are nonnegative integers not exceeding \( n \) and
\[ E(u, p) = (1 - u) \exp [u + (1/2)u^2 + \cdots + (1/p)u^p] \]

if \( p > 0 \) whereas \( E(u, 0) = (1 - u) \). Taking the logarithmic derivative of (2.1) and simplifying, one finds that

\[ \frac{f''(z)}{f(z)} = \frac{m}{z} + P'(z) + A(z) - B(z) \]

where

\[ A(z) = \sum_{i=1}^{\infty} \frac{z^p}{a^*_j(z - a_j)} \quad B(z) = \sum_{j=1}^{\infty} \frac{z^q}{b^*_j(z - b_j)}. \]

By hypothesis, \( f'(^k) = 0, \ k = 1, 2, \cdots, n. \) Hence, from (2.2), follows that for \( k = 1, 2, \cdots, n \)

\[ P'(^k) = -(m/^k) - A(^k) + B(^k). \]

Since \( P'(z) \) is a polynomial of degree at most \( n - 1 \), it can be represented by the Lagrange Interpolation Formula as

\[ \frac{P'(z)}{\psi(z)} = \sum_{k=1}^{n} \frac{P'(^k)}{\psi'(^k)(z - ^k)}. \]

Hence, using (2.3) and (2.4), one finds that

\[ \frac{P'(z)}{\psi(z)} = - \sum_{k=1}^{n} \frac{m}{\zeta_k^p \psi'(^k)(z - ^k)} - \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{\zeta_k^p}{a^*_j \psi'(^k)(z - ^k)(^k - a_j)} \]

\[ + \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{\zeta_k^q}{b^*_j \psi'(^k)(z - ^k)(^k - b_j)}. \]

In view of the fact that sums \( A(z) \) and \( B(z) \) are uniformly and absolutely convergent on every compact set that omits all the \( a_i \) and \( b_j \), the order of summation of the double sums in (2.5) can be reversed. Thus the first double sum in (2.5) becomes

\[ \sum_{j=1}^{\infty} \frac{1}{a_j^p} \sum_{k=1}^{n} \frac{\zeta_k^p}{\psi'(^k)(z - ^k)} = \sum_{j=1}^{\infty} \frac{1}{a_j^p (z - a_j)} \left[ S(z) - S(a_j) \right] \]

where

\[ S(z) = \psi(z) \sum_{k=1}^{n} \frac{\zeta_k^p}{(z - ^k)(\psi'(^k))}. \]

Since the polynomial \( S(z) \) is of degree at most \( n - 1 \) with \( S(^k) = \zeta_k^p \), the polynomial

\[ T(z) = S(z) - z^p \]

is of degree at most \( n \) such that
Therefore \( T(\zeta_k) = 0 \), for \( k = 1, 2, \ldots, n \).

Therefore \( T(z) = c \psi(z) \), where \( c \) is a constant that may be zero. Accordingly,

\[
S(z) = z^p + c \psi(z)
\]

and the sum (2.6) becomes

\[
\sum_{j=1}^{\infty} \frac{z^p}{\psi(z)\alpha_j^p(z - a_j)} - \sum_{j=1}^{\infty} \frac{1}{\psi'(a_j)(z - a_j)}.
\]

Similarly the second double sum in (2.5) reduces to

\[
\sum_{j=-1}^{\infty} \frac{z^q}{\psi(z)b_j^q(z - b_j)} - \sum_{j=-1}^{\infty} \frac{1}{\psi(b_j)(z - b_j)}.
\]

Finally, on use of the Lagrange Interpolation Formula for \( 1/\psi(z) \), the single sum in (2.5) becomes

\[
\frac{m}{z} \sum_{k=1}^{n} \left[ \frac{1}{\zeta_k} + \frac{1}{z - \zeta_k} \right] \frac{1}{\psi'(\zeta_k)} = \frac{m}{z} \left[ -\frac{1}{\psi(0)} + \frac{1}{\psi(z)} \right].
\]

Substituting from (2.9), (2.10) and (2.11) into (2.5), one reduces (2.2) to

\[
f'(z) = \frac{m \psi(z)}{z \psi(0)} + \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(a_j)(z - a_j)} - \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(b_j)(z - b_j)}.
\]

However, the first term here may be dropped since it is obtainable from the first or second sum in (2.12) by allowing either \( m a_j \) (if \( m > 0 \)) or \(-m b_j \) (if \( m < 0 \)) to coalesce at 0. Thus identity (1.1) has been established.

3. Location of critical points. An immediate consequence of Theorem 1.1 is the following:

**Theorem 3.1.** Let \( f \) be a meromorphic function of finite order \( \rho \) possessing the finite zeros \( a_1, a_2, a_3, \ldots \) and poles \( b_1, b_2, b_3, \ldots \) and let \( \zeta_0, \zeta_1, \ldots, \zeta_n \), be any \( n + 1 = [\rho] + 1 \) distinct critical points of \( f \) which are not also zeros of \( f \). Then

\[
\sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - a_j)(\zeta_1 - a_j) \cdots (\zeta_n - a_j)} = \sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - b_j)(\zeta_1 - b_j) \cdots (\zeta_n - b_j)}.
\]

Equation (3.1) follows from (1.1) on setting \( z = \zeta_0 \), writing out \( \psi'(a_j) \) according to (1.2) and cancelling the factor \( \psi'(\zeta_0) \neq 0 \).

As an application of (3.1), the following will now be proved.
Theorem 3.2. Let $D_a$ and $D_b$ be two regions with which can be associated a set $R$ of points $\zeta$ such that a ray from $\zeta$ to some point $\gamma$ separates $\bar{D}_a$ from $\bar{D}_b$ and such that inequality
\[
0 < \arg \left( (\gamma - \zeta)/(z - \zeta) \right) < \pi/(n + 1) \pmod{2\pi}
\]
holds for all $z$ in one of the regions $D_a$, $D_b$ and inequality
\[
-\pi/(n + 1) < \arg \left( (\gamma - \zeta)/(z - \zeta) \right) < 0 \pmod{2\pi}
\]
holds for all $z$ in the other region. Let $f$, a meromorphic function of finite order $\rho$, have all its zeros in $D_a$ and all its poles in $D_b$. Then at most $n = \lfloor \rho \rfloor$ critical points of $f$ lie in $R$.

Proof. If on the contrary $n+1$ distinct critical points $\zeta_0, \zeta_1, \ldots, \zeta_n$ were in $R$, identity (3.1) holds for them in relation to the zeros and poles of $f$. By hypothesis, one can associate with each $\zeta_k$, a point $\gamma_k$ such that for $j = 1, 2, 3, \ldots$ the inequalities
\[
\begin{align*}
(3.2) \quad & 0 < \arg \frac{\gamma_k - \zeta_k}{a_j - \zeta_k} < \frac{\pi}{n + 1} \pmod{2\pi} \\
(3.3) \quad & -\frac{\pi}{n + 1} < \arg \frac{\gamma_k - \zeta_k}{b_j - \zeta_k} < 0 \pmod{2\pi}
\end{align*}
\]
hold (or those with $a_j$ and $b_j$ interchanged). Setting
\[
T(z) = \prod_{k=0}^{n} \left[ (\gamma_k - \zeta_k)/(z - \zeta_k) \right],
\]
one infers that
\[
0 < \arg T(a_j) < \pi, \quad -\pi < \arg T(b_j) < 0,
\]
for all $j$. This means that
\[
0 < \arg \sum_{j=1}^{\infty} T(a_j) < \pi, \quad -\pi < \arg \sum_{j=1}^{\infty} T(b_j) < 0.
\]
Consequently,
\[
\sum_{j=1}^{\infty} T(a_j) \neq \sum_{j=1}^{\infty} T(b_j),
\]
in contradiction to (3.1). Consequently, at most $n$ distinct critical points $\zeta_k$ can lie on $R$, as was to be proved.

As an illustration, let $f$ be a meromorphic function of order $\rho$,
$1 \leq \rho < \infty$, and let

$$D_a = \{z = x + iy: x > 0, \quad 0 \leq h < y < x \tan \left[\frac{\pi}{(n + 1)}\right]\}$$

$$D_b = \{z = x + iy: x > 0, \quad -h > y > -x \tan \left[\frac{\pi}{(n + 1)}\right]\}.$$ 

Then, according to Theorem 3.2, at most $n$ critical points of $f$ lie in the region

$$R = \{z = x + iy: x < 0, \quad |y| < \min [h, |x| \tan \left[\frac{\pi}{(n + 1)}\right]]\}$$

**Remark.** In identity (1.1) and the subsequent theorems, $f'(z)$ may be replaced by the linear combination $f'(z) + \lambda f(z)$ or more generally by

$$F_1(z) = f'(z) + f(z)g'(z)$$

where $g(z)$ is an arbitrary polynomial of degree at most $n$, provided $\zeta_0, \zeta_1, \zeta_2, \ldots \zeta_n$ are taken as the zeros of $F_1(z)$. This follows from the fact that the meromorphic function $F(z) = e^{\phi(z)}f(z)$ is also of order $\rho$, has the same zeros and poles as $f$ and $F''(z) = e^{\phi(z)}F_1(z)$.

**Reference**


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