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## **EXAMPLES CONCERNING SUM PROPERTIES FOR METRIC-DEPENDENT DIMENSION FUNCTIONS**

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## EXAMPLES CONCERNING SUM PROPERTIES FOR METRIC-DEPENDENT DIMENSION FUNCTIONS

Dedicated to Professor J. H. Roberts on the  
occasion of his sixty-fifth birthday.

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Let  $d_0$  denote the metric dimension function defined by Katětov, and let  $\dim$  be the covering dimension function. K. Nagami and J. H. Roberts introduced the metric-dependent dimension functions  $d_2$  and  $d_3$ , and J. C. Smith defined the functions  $d_6$  and  $d_7$ . The following relations hold for all metric spaces  $(X, \rho)$ :

$$d_2(X, \rho) \leq d_3(X, \rho) \leq d_6(X, \rho) \leq d_7(X, \rho) \leq d_0(X, \rho).$$

Since all of the metric-dependent dimension functions above satisfy a "Weak Sum Theorem," it is natural to ask if any of these functions satisfy the Finite Sum Theorem or the Countable Sum Theorem. In this paper the authors obtain new properties of these dimension functions, and using these results construct examples for which none of the metric dependent dimension functions satisfy either of the sum theorems in question.

Let  $d_0$  denote the metric dimension function defined by Katětov [2], and let  $\dim$  be the covering dimension function. K. Nagami and J. H. Roberts [5] introduced the metric-dependent dimension functions  $d_2$  and  $d_3$ , and J. C. Smith [7] defined the functions  $d_6$  and  $d_7$ . The following relations hold for all metric spaces  $(X, \rho)$ :

$$(*) \quad d_2(X, \rho) \leq d_3(X, \rho) \leq d_6(X, \rho) \leq d_7(X, \rho) \leq d_0(X, \rho).$$

In [8] J. C. Smith has shown that all of the above dimension functions satisfy the "Weak Sum Theorem" stated below for  $d_2$ .

**THEOREM.** *Let  $(X, \rho)$  be a metric space satisfying these conditions:*

- (1)  $X = \bigcup_{\alpha \in A} F_\alpha$ , where each  $F_\alpha$  is closed in  $X$ .
- (2)  $\{F_\alpha: \alpha \in A\}$  is locally finite.
- (3)  $d_2(F_\alpha, \rho) \leq n$  for each  $\alpha \in A$ .
- (4)  $\dim[(\text{bdry } F_\alpha) \cap F_\beta] \leq n - 1$  for  $\alpha \neq \beta$ .

Then  $d_2(X, \rho) \leq n$ .

It is now natural to ask the following question. Do any of the above dimension functions satisfy the Countable Sum Theorem or the Finite Sum Theorem? In this paper we answer this question in the

negative. In § 2 we obtain a number of results relating the dimension functions  $d_2$  and  $d_0$  to certain subsets of Euclidean  $n$ -space. In § 3 we apply these results in constructing a metric space for which none of the above metric-dependent dimension functions satisfies the Finite Sum Theorem.<sup>1</sup> In § 4 we prove that if any countable disjoint collection of compact subsets of Euclidean  $n$ -space ( $n \geq 3$ ) is removed, the dimension function  $d_2$  may decrease by at most 1. This result is analogous to Theorem 1 of [5]. As an application of this theorem we construct an example of a metric space for which none of the above metric-dependent dimension functions satisfies the Countable Sum Theorem.

## 2. Definitions and preliminary results.

DEFINITION 2.1. Let  $(X, \rho)$  be a metric space and let  $B, C$  and  $D$  be closed subsets of  $X$ . The set  $B$  is said to *separate*  $C$  and  $D$  in  $X$  if  $X - B = S \cup T$  where  $S$  and  $T$  are nonempty open sets,  $C \subseteq S$  and  $D \subseteq T$ .

DEFINITION 2.2. Let  $(X, \rho)$  be a nonempty metric space and let  $n$  be a nonnegative integer. Then  $d_2(X, \rho) \leq n$  if  $(X, \rho)$  satisfies this condition:

( $D_2$ ) For any collection  $\mathcal{C} = \{(C_i, C'_i) : i = 1, \dots, n + 1\}$  of  $n + 1$  pairs of closed sets with  $\rho(C_i, C'_i) > 0$  for each  $i = 1, \dots, n + 1$ , there exist closed sets  $B_i, i = 1, \dots, n + 1$ , such that

(i)  $B_i$  separates  $C_i$  and  $C'_i$  for each  $i = 1, \dots, n + 1$  and

(ii)  $\bigcap_{i=1}^{n+1} B_i = \emptyset$ .

If  $X = \emptyset$  then  $d_2(X, \rho) = -1$ .

DEFINITION 2.3. Let  $(X, \rho)$  be a nonempty metric space. The *metric dimension* of  $(X, \rho)$ , written  $d_0(X, \rho)$ , is the smallest integer  $k$  such that for every  $\varepsilon > 0$  there exists an open cover  $\mathcal{U}$  of  $X$  with  $\text{mesh}(\mathcal{U}) < \varepsilon$  and  $\text{order}(\mathcal{U}) \leq k + 1$ .

DEFINITION 2.4. Let  $d$  denote a dimension function on the class of all metric spaces. Then  $d$  is said to have the *Finite Sum Property* if given any metric space  $(X, \rho)$  which is the finite union of closed subsets  $A_i$  with  $d(A_i, \rho) \leq n$ , then  $d(X, \rho) \leq n$ . Also  $d$  is said to have the *Monotone Sum Property* if given any metric space  $(X, \rho)$  which is the countable union of closed subspaces  $A_i$  with  $A_i \subseteq A_{i+1}$  and  $d(A_i, \rho) \leq n$  for each  $i$ , then  $d(X, \rho) \leq n$ .

DEFINITION 2.5. Let  $(X, \rho)$  be a metric space. Suppose

<sup>1</sup> The basic idea for Example 3.1 is due to J.H. Roberts.

$$\mathcal{C} = \{(C_i, C'_i): i = 1, \dots, n\}$$

is a collection of  $n$  pairs of disjoint closed subsets of  $X$  with the property that, if  $B_i$  is a closed set separating  $C_i$  and  $C'_i$  in  $X$  for each  $i$ , then  $\bigcap_{i=1}^n B_i \neq \emptyset$ . Then  $\mathcal{C}$  will be called an  $n$ -defining system for  $X$ .

**DEFINITION 2.6.** A decomposition of a metric space  $(X, \rho)$ ,  $X = \bigcup_{\alpha \in A} H_\alpha$  will be called a *proper decomposition* if there are at least two indices  $\alpha, \beta \in A$  such that  $H_\alpha \neq \emptyset$  and  $H_\beta \neq \emptyset$ .

The following characterization of  $d_0$  is proved in [5] page 426.

**LEMMA 2.7.** Let  $(X, \rho)$  be a metric space. Then  $d_0(X, \rho) \leq n$  if and only if there exists a sequence of locally finite closed coverings  $\{\mathcal{F}_i: i \geq 1\}$  such that

- (i)  $\text{mesh}(\mathcal{F}_i) < 1/i$  for each  $i$ .
- (ii)  $\text{ord}(\mathcal{F}_i) \leq n + 1$  for each  $i$ .

The following theorem is proved in [4].

**THEOREM 2.8.** Let  $X$  be a metric space with  $\dim X \leq n$ , and let  $B_0, B_1, \dots$  be a sequence of closed subsets of  $X$  such that  $B_0 = X$  and  $\dim(B_i) = n_i$ . Let  $\varepsilon > 0$ . Then there exists a locally finite closed covering  $\mathcal{F} = \{F_\alpha: \alpha \in \Gamma\}$  which satisfies the following conditions:

- (i)  $\text{mesh}(\mathcal{F}) < (\varepsilon)$ .
- (ii) for each  $i$ ,  $\text{ord}(\mathcal{F} | B_i) \leq n_i + 1$ .
- (iii) for each  $i$  and for each  $j \leq n_i + 2$ ,  $\dim \bigcap_{k=1}^j [F_{\alpha(k)} \cap B_i] \leq n_i - j + 1$ , where  $\{\alpha(1), \alpha(2), \dots, \alpha(j)\}$  is any collection of  $j$  distinct members of  $\Gamma$ .

**THEOREM 2.9.** Let  $X$  be a metric space with  $\dim X \leq n$ , and let  $B_0, B_1, \dots$  be a sequence of closed subsets of  $X$  with  $\dim(B_i) = n_i$  and  $B_0 = X$ . Let  $\varepsilon > 0$ . Then there exists a locally finite closed covering  $\mathcal{F} = \{F_\alpha: \alpha \in \Gamma\}$  which satisfies the following conditions:

- (i)  $\text{mesh}(\mathcal{F}) < \varepsilon$ .
- (ii) if  $A_j = \{x: \text{ord}(x, \mathcal{F}) \geq j\}$  for  $j = 1, \dots, n + 1$ , then

$$\dim(A_j \cap B_i) \leq n_i - j + 1$$

for each  $i$ .

*Proof.* By Theorem 2.8 above there exists a locally finite closed cover  $\mathcal{F} = \{F_\alpha: \alpha \in \Gamma\}$  which satisfies (i)-(iii) of that theorem. By (ii) with  $B_0 = X$ , we have  $\text{ord}(\mathcal{F}) \leq n + 1$ . For each  $k$ ,  $1 \leq k \leq n + 1$ , define  $H_k = \{x: \text{ord}(x, \mathcal{F}) = k\}$  so that  $A_j = \bigcup_{k=j}^{n+1} H_k$ . Let  $\Gamma_k$

be the collection of all subsets of  $\Gamma$  whose cardinality is  $k$ , and let  $\mathcal{L}_k = \{\bigcap_{\alpha \in g} F_\alpha : g \in \Gamma_k\}$ . Since  $\mathcal{F}$  is locally finite,  $\mathcal{L}_k$  is a locally finite collection of closed sets whose union is  $A_k$ . Thus by (iii) in Theorem 2.8 we have that  $\dim(L \cap B_i) \leq n_i - j + 1$  for each  $L \in \mathcal{L}_k$ . Hence by the Locally Finite Sum Theorem for covering dimension,  $\dim(A_j \cap B_i) \leq n_i - j + 1$  for each  $i$ .

As an application of Theorem 2.9 above we obtain the following.

**THEOREM 2.10.** *Let  $\{J_i : i \geq 1\}$  be a sequence of closed subsets of a Cantor 3-manifold  $(K, \rho)$  such that  $\dim(J_i) \leq 1$  for each  $i$ . Then there exists a sequence of closed sets  $\{H_i : i \geq 1\}$  satisfying the following properties:*

- (1) For each  $i$ ,  $H_i \subseteq K - \bigcup_{k \geq 1} J_k$ .
- (2)  $H_i \cap H_j = \emptyset$  for  $i \neq j$ .
- (3)  $\dim(H_i) \leq 1$  for each  $i$ .
- (4)  $d_0(K - [(\bigcup_{i \geq 1} J_i) \cup (\bigcup_{i \geq 1} H_i)]) \leq 1$ .

*Proof.* Since  $(K, \rho)$  is a Cantor 3-manifold there exists a 3-defining system  $\mathcal{C}_3 = \{(C_i, C'_i) : i = 1, 2, 3\}$  for  $(K, \rho)$ . We may assume that  $\rho(C_i, C'_i) > \delta > 0$ . By Theorem 2.9 with  $B_i = J_i$  for  $i \geq 1$ , there exists a finite closed cover  $\mathcal{F}_1$  of  $K$  satisfying,

- (1)  $\text{mesh}(\mathcal{F}_1) < \delta$
- (2) if  $A_j^1 = \{x : \text{ord}(x, \mathcal{F}_1) \geq j\}$  for  $j = 1, \dots, 4$  then

$$\dim(A_j^1 \cap B_0) \leq 3 - j + 1.$$

Let  $H_0 = B_0$  and  $H_1 = A_3^1$ . By (2) above we have  $\dim(A_3^1) \leq 1$ . By Theorem 1 in [6] we have  $\dim(A_3^1) \geq 1$ . Hence  $\dim(A_3^1) = 1$ . Therefore by Theorem 2.9,  $\dim(H_1 \cap J_i) \leq 1 - 3 + 1 = -1$ , so that  $H_1 \cap J_i = \emptyset$  for all  $i \geq 1$ .

We apply Theorem 2.9 again with  $B_i = H_i$  for  $i = 0, 1$  and  $B_i = J_{i-1}$  for  $i \geq 2$ , and  $\varepsilon = \delta/2$ . Thus there exists a finite closed cover  $\mathcal{F}_2$  of  $K$  satisfying.

- (1)  $\text{mesh}(\mathcal{F}_2) < \delta/2$ .
- (2) if  $A_j^2 = \{x : \text{ord}(x, \mathcal{F}_2) \geq j\}$  for  $j = 1, \dots, 4$  then

$$\dim(A_j^2 \cap H_i) \leq n_i - j + 1 \quad \text{for } i = 0, 1.$$

Let  $H_2 = A_3^2$ . By the same argument as above we have  $\dim(H_2) = 1$  and  $\dim(H_1 \cap H_2) \leq -1$ , so that  $H_1 \cap H_2 = \emptyset$ . Similarly  $H_i \cap J_k = \emptyset$  for  $i = 1, 2$  and any  $j \geq 1$ .

Repeating this process we obtain a sequence of finite closed covers  $\{\mathcal{F}_i : i \geq 1\}$  satisfying the following:

- (1)  $\text{mesh}(\mathcal{F}_i) < \delta/i$  for each  $i \geq 1$ .
- (2) with  $H_i = A_3^i$  for each  $i \geq 1$ , we have  $\dim(H_i) = 1$ , and  $H_i$

is a closed subset of  $K$ .

(3)  $H_i \cap H_j = \emptyset$  for all  $i \neq j$ , and  $H_i \cap J_k = \emptyset$  for all  $i \geq 1, k \geq 1$ .

Let  $X = K - [(\bigcup_{i \geq 1} H_i) \cup (\bigcup_{i \geq 1} J_i)]$ . Then  $d_0(X, \rho) \leq 1$  by Lemma 2.7.

The following Lemma is proved in [3, p. 21]

LEMMA 2.11. *Let  $X$  be a compact metric space with disjoint closed subsets  $D$  and  $E$ . Then one of the following must be true.*

(1) *There exists a continuum  $W$  in  $X$  such that  $W \cap D \neq \emptyset$  and  $W \cap E \neq \emptyset$ .*

(2) *The sets  $D$  and  $E$  can be separated by the empty set.*

The following is an immediate consequence of Lemma 2.11.

LEMMA 2.12. *Let  $(X, \rho)$  be a compact metric space, let*

$$\mathcal{C} = \{(C_i, C'_i) : i = 1, \dots, n\}$$

*be an  $n$ -defining system for  $X$ , and let  $Y \subseteq X$  such that*

$$d_2(X - Y) \leq n - 2.$$

*If for each  $i = 1, \dots, n - 1, B_i$  is a closed set separating  $C_i$  from  $C'_i$ , then there exists a continuum  $W \subseteq \bigcap_{i=1}^{n-1} B_i \subseteq Y$  such that  $W \cap C_n \neq \emptyset$  and  $W \cap C'_n \neq \emptyset$ .*

The following is proved in [5, p. 416]

LEMMA 2.13. *If  $X$  is a connected compact Hausdorff space then there is no countable proper decomposition of  $X$  into mutually disjoint closed subsets.*

The following is an easy consequence of Lemma 2.13.

LEMMA 2.14. *If  $(X, \rho)$  is a closed connected subset of Euclidean  $n$ -space, then there is no proper decomposition of  $X$  into a countable collection of mutually disjoint, compact sets.*

**3. No metric-dependent dimension function has the finite sum property.**

EXAMPLE 3.1. Using Theorem 2.10 we construct a metric space  $(X, \rho)$  with the property that  $X = A_1 \cup A_2$ , where

$$d_0(A_1, \rho) \leq 1, d_0(A_2, \rho) \leq 1$$

but  $d_2(X, \rho) \geq 2$ . Hence by the relation (\*) above  $(X, \rho)$  is an example for which none of the metric-dependent dimension functions have the Finite Sum Property.

Let  $Y_1 = \{(x_1, x_2, x_3) : 0 \leq x_i \leq 1 \ i = 1, 2, 3\}$ . Since  $(Y_1, \rho)$  is a Cantor 3-manifold by Theorem 2.10, with  $J_i = \emptyset$  for all  $i \geq 1$ , there exists a sequence of sets  $\{H_i : i \geq 1\}$  satisfying:

- (1) each  $H_i$  is closed in  $Y_1$ .
- (2)  $H_i \cap H_j = \emptyset$  for  $i \neq j$ .
- (3)  $\dim(H_i) \leq 1$  for all  $i$ .
- (4)  $d_0(Y_1 - \bigcup_{i \geq 1} H_i) \leq 1$ .

Similarly let  $Y_2 = \{(x_1, x_2, x_3) : 0 \leq x_i \leq 1 \text{ for } i = 2, 3 \text{ and } 1 \leq x_1 \leq 2\}$ . Again by Theorem 2.10 with  $J_i = H_i$  for each  $i \geq 1$ , there is a sequence of sets  $\{L_i : i \geq 1\}$  with the following properties:

- (5) each  $L_i$  is closed in  $Y_2$  and  $L_i \subseteq Y_2 - \bigcup_{k \geq 1} H_k$ .
- (6)  $L_i \cap L_j = \emptyset$  for  $i \neq j$ .
- (7)  $\dim(L_i) \leq 1$  for all  $i \geq 1$ .
- (8)  $d_0(Y_2 - (\bigcup_{k \geq 1} H_k \cup \bigcup_{k \geq 1} L_k)) \leq 1$ .

Let  $A_1 = Y_1 - (\bigcup_{i \geq 1} H_i \cup \bigcup_{i \geq 1} L_i)$ , let  $A_2 = Y_2 - (\bigcup_{i \geq 1} H_i \cup \bigcup_{i \geq 1} L_i)$  and define  $X = A_1 \cup A_2$ . Then  $d_0(A_1, \rho) \leq 1$  and  $d_0(A_2, \rho) \leq 1$  by (4) and (8) above. We assert that  $d_2(X, \rho) \geq 2$ . Suppose  $d_2(X, \rho) \leq 1$ . Let

$$C_1 = \{(x_1, x_2, x_3) : x_1 = 0; 0 \leq x_i \leq 1, i = 1, 2\}$$

$$C'_1 = \{(x_1, x_2, x_3) : x_1 = 2; 0 \leq x_i \leq 1, i = 1, 2\}$$

and let

$$C_i = \{(x_1, x_2, x_3) : 0 \leq x_1 \leq 2; x_i = 0; 0 \leq x_j \leq 1, j \neq i\}$$

$$C'_i = \{(x_1, x_2, x_3) : 0 \leq x_1 \leq 2; x_i = 1; 0 \leq x_j \leq 1, j \neq i\} \quad \text{for } i = 2, 3.$$

Then  $\mathcal{C} = \{(C_i, C'_i) : i = 1, 2, 3\}$  is a 3-defining system for the compact metric space  $Z = Y_1 \cup Y_2$ . By Lemma 2.12 there exists a continuum  $G \subseteq (\bigcup_{i \geq 1} H_i) \cup (\bigcup_{i \geq 1} L_i)$  such that  $G \cap C_1 \neq \emptyset$  and  $G \cap C'_1 \neq \emptyset$ . Since  $H_i \cap C'_1 = \emptyset$  and  $L_i \cap C_1 = \emptyset$  for each  $i$ ,  $G$  is not contained in any one  $H_i$  or  $L_i$ . Define  $M_{2i} = G \cap H_i$  and  $M_{2i-1} = G \cap L_i$  for each  $i \geq 1$ . Then  $G = \bigcup_{i \geq 1} M_i$  and  $M_i \cap M_j = \emptyset$  for  $i \neq j$ . The collection  $\{M_i : i \geq 1\}$  is thus a proper decomposition of  $G$ . This contradicts Lemma 2.13 above. Therefore  $d_2(X, \rho) \geq 2$ .

**4. No metric-dependent dimension function has the monotone sum property.**

**THEOREM 4.1.** *Let  $(E^n, \rho)$  denote Euclidean  $n$ -space for  $n \geq 3$ . Let  $\{A_i : i \geq 1\}$  be any collection of compact subsets of  $E^n$  such that*

$A_i \cap A_j = \emptyset$  for all  $i \neq j$ . Then  $d_2(E^n - \bigcup_{i \geq 1} A_i) \geq n - 1$ .

*Proof.* Suppose  $d_2(E^n - \bigcup_{i \geq 1} A_i) \leq n - 2$ . For each  $i = 1, \dots, n - 1$  let  $C_i = \{(x_1, \dots, x_n); x_i = 1\}$  and let  $C'_i = \{(x_1, \dots, x_n); x_i = -1\}$ . For each  $j \geq 1$ , define  $S_j = \{(x_1, \dots, x_n); x_n = j\}$  and let

$$S_0 = \{(x_1, \dots, x_n); x_n = 0, 0 \leq x_i \leq 1, i = 1, \dots, n - 1\}.$$

Then for any  $j \geq 1$  the collection

$$\mathcal{C}_j = \{(C_i, C'_i); i = 1, \dots, n - 1\} \cup (S_j, S_0)$$

is an  $n$ -defining system for the compact space

$$T_j = \{(x_1, \dots, x_n); |x_i| \leq j\}.$$

Since  $d_2(E^n - \bigcup_{i \geq 1} A_i) \leq n - 2$ , there exist closed sets  $B_1, \dots, B_{n-1}$  such that  $B_i$  separates  $C_i$  from  $C'_i$  for each  $i = 1, \dots, n - 1$ , and  $B = \bigcap_{i=1}^{n-1} B_i \subseteq \bigcup_{i \geq 1} A_i$ . By Lemma 2.12, for each  $j \geq 1$  there exists a continuum  $D_j$  such that  $D_j \subseteq B \subseteq \bigcup_{i \geq 1} A_i$ ,  $D_j \cap S_0 \neq \emptyset$  and  $D_j \cap S_j \neq \emptyset$ . If  $0 < k < j$ , then  $S_k$  separates  $S_0$  from  $S_j$  in  $T$ . Thus for all  $j \geq 1$  and all  $k$  satisfying  $0 < k \leq j$ , we have that  $D_j \cap S_k \neq \emptyset$ .

We have thus proved the statement:

(1) For each  $j \geq 1$ ,  $B$  contains the continuum  $D_j$  such that  $D_j \cap S_k \neq \emptyset$  for all  $k$ ,  $0 \leq k \leq j$ .

Since  $\liminf \{D_j; j \geq 1\} \neq \emptyset$  we have by [1, p. 100] that in any  $T_j$ ,  $R = \limsup \{D_j; j \geq 1\}$  is connected. Note that  $R \subseteq B$  since  $B$  is closed. From statement (1) above we now have,

(2)  $R$  is a connected set with the property that  $R \cap S_j \neq \emptyset$  for every  $j \geq 1$ .

Since each  $A_i$  is compact,  $R$  cannot be contained in any one  $A_i$ . Let  $H_i = A_i \cap R$ . Then  $\{H_i; i \geq 1\}$  is a proper decomposition of the connected set  $R$  into a collection of mutually disjoint compact sets. This contradicts Lemma 2.14 and completes the proof of the theorem.

**EXAMPLE 4.2.** We construct a metric space  $(X, \rho)$  with the property that  $X = \bigcup_{i \geq 1} A_i$ , where for each  $i \geq 1$ ,  $A_i$  is a closed set,  $d_0(A_i, \rho) \leq 1$  and  $A_i \subseteq A_{i+1}$ , yet  $d_2(X, \rho) \geq 2$ . For each  $i \geq 1$ , let  $T_i = \{(x_1, x_2, x_3); |x_i| \leq i\}$ . Then  $\bigcup_{i \geq 1} T_i = E^3$  and  $T_i \subseteq T_{i+1}$  for all  $i \geq 1$ . For each  $i \geq 1$  we construct a sequence of closed subsets  $\{H_{ik}; k \geq 1\}$  of  $T_i$ . Applying Theorem 2.10 to the Cantor 3-manifold  $T_1$ , with  $J_k = \emptyset$  for all  $k \geq 1$ , we obtain a sequence of closed subsets

$$\{H_{1k}; k \geq 1\}$$

of  $T_1$  such that:

(1)  $H_{1k} \cap H_{1j} = \emptyset$  for all  $j \neq k$ .



(2)  $\dim(H_{1k}) \leq 1$  for all  $k \geq 1$ .

(3)  $d_0(T_1 - \bigcup_{k \geq 1} H_{1k}) \leq 1$ .

Suppose that for each  $i = 1, \dots, m$  the closed collection  $\{H_{ik}: k \geq 1\}$  has been constructed satisfying the following,

(1)  $H_{ik} \cap H_{lj} = \emptyset$  for  $(i, k) \neq (l, j); 1 \leq i \leq m, 1 \leq l \leq m$ .

(2)  $\dim(H_{ik}) \leq 1$  for all  $i, 1 \leq i \leq m$ , and for all  $k \geq 1$ .

(3)  $d_0(T_i - \bigcup_{j=1}^i [\bigcup_{k \geq 1} H_{jk}]) \leq 1$ .

To construct the collection  $\{H_{m+1,k}: k \geq 1\}$  we apply Theorem 2.10 again to the Cantor 3-manifold  $T_{m+1}$ , identifying the collection  $\{J_i: i \geq 1\}$  in the theorem with the collection  $\{H_{jk}: j = 1, \dots, m; k \geq 1\}$ . Finally we conclude that there exists a countable collection of compact sets  $\{H_{ik}: i \geq 1, k \geq 1\}$  satisfying the following:

(1)  $H_{ik} \cap H_{lj} = \emptyset$  for all  $(i, k) \neq (l, j)$

(2)  $\dim(H_{ik}) \leq 1$  for all  $i, k$ .

(3)  $d_0(T_i - \bigcup_{j=1}^i [\bigcup_{k \geq 1} H_{jk}]) \leq 1$ , for all  $i$ .

For each  $i \geq 1$ , we define  $W_i = T_i - \bigcup_{j=1}^i [\bigcup_{k \geq 1} H_{jk}]$ ,

$$A_i = W_i - \bigcup_{j=1}^{\infty} \left[ \bigcup_{k \geq 1} H_{jk} \right],$$

and  $X = \bigcup_{i \geq 1} A_i$ . Now  $X$  is a monotone sum since for all  $i \geq 1$ ,  $A_i \subseteq A_{i+1}$ , and  $A_i$  is closed in  $X$ . Also we have that

$$X = E^3 - \bigcup_{j=1}^{\infty} \left[ \bigcup_{k=1}^{\infty} H_{jk} \right]$$

and for each  $i$ ,  $d_0(W_i) \leq 1$  by construction. Therefore  $d_0(A_i) \leq 1$  since  $A_i \subseteq W_i$  for each  $i$ .

Now the collection  $\{H_{jk}: j \geq 1, k \geq 1\}$  is a countable collection of compact mutually disjoint subsets of  $E^3$ . Hence by Theorem 4.1 above we have that  $d_2(X) \geq 2$ . Thus  $(X, \rho)$  is an example of a metric space for which none of the metric-dependent dimension functions satisfy the Monotone Sum Property.

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Bruce Alan Barnes, <i>Banach algebras which are ideals in a Banach algebra</i> . . . . .	1
David W. Boyd, <i>Inequalities for positive integral operators</i> . . . . .	9
Lawrence Gerald Brown, <i>Note on the open mapping theorem</i> . . . . .	25
Stephen Daniel Comer, <i>Representations by algebras of sections over Boolean spaces</i> . . . . .	29
John R. Edwards and Stanley G. Wayment, <i>On the nonequivalence of conservative Hausdorff methods and Hausdorff moment sequences</i> . . . . .	39
P. D. T. A. Elliott, <i>On the limiting distribution of additive functions (mod 1)</i> . . . . .	49
Mary Rodriguez Embry, <i>Classifying special operators by means of subsets associated with the numerical range</i> . . . . .	61
Darald Joe Hartfiel, <i>Counterexamples to a conjecture of G. N. de Oliveira</i> . . . . .	67
C. Ward Henson, <i>A family of countable homogeneous graphs</i> . . . . .	69
Satoru Igari and Shigehiko Kuratsubo, <i>A sufficient condition for <math>L^p</math>-multipliers</i> . . . . .	85
William A. Kirk, <i>Fixed point theorems for nonlinear nonexpansive and generalized contraction mappings</i> . . . . .	89
Erwin Kleinfeld, <i>A generalization of commutative and associative rings</i> . . . . .	95
D. B. Lahiri, <i>Some restricted partition functions. Congruences modulo 11</i> . . . . .	103
T. Y. Lin, <i>Homological algebra of stable homotopy ring <math>\pi_*</math> of spheres</i> . . . . .	117
Morris Marden, <i>A representation for the logarithmic derivative of a meromorphic function</i> . . . . .	145
John Charles Nichols and James C. Smith, <i>Examples concerning sum properties for metric-dependent dimension functions</i> . . . . .	151
Asit Baran Raha, <i>On completely Hausdorff-completion of a completely Hausdorff space</i> . . . . .	161
M. Rajagopalan and Bertram Manuel Schreiber, <i>Ergodic automorphisms and affine transformations of locally compact groups</i> . . . . .	167
N. V. Rao and Ashoke Kumar Roy, <i>Linear isometries of some function spaces</i> . . . . .	177
William Francis Reynolds, <i>Blocks and <math>F</math>-class algebras of finite groups</i> . . . . .	193
Richard Rochberg, <i>Which linear maps of the disk algebra are multiplicative</i> . . . . .	207
Gary Sampson, <i>Sharp estimates of convolution transforms in terms of decreasing functions</i> . . . . .	213
Stephen Scheinberg, <i>Fatou's lemma in normed linear spaces</i> . . . . .	233
Ken Shaw, <i>Whittaker constants for entire functions of several complex variables</i> . . . . .	239
James DeWitt Stein, <i>Two uniform boundedness theorems</i> . . . . .	251
Li Pi Su, <i>Homomorphisms of near-rings of continuous functions</i> . . . . .	261
Stephen Willard, <i>Functionally compact spaces, <math>C</math>-compact spaces and mappings of minimal Hausdorff spaces</i> . . . . .	267
James Patrick Williams, <i>On the range of a derivation</i> . . . . .	273