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EXAMPLES CONCERNING SUM PROPERTIES FOR METRIC-DEPENDENT DIMENSION FUNCTIONS

Dedicated to Professor J. H. Roberts on the occasion of his sixty-fifth birthday.

J. C. NICHOLS AND J. C. SMITH

Let d_0 denote the metric dimension function defined by Katětov, and let dim be the covering dimension function. K. Nagami and J. H. Roberts introduced the metric-dependent dimension functions d_2 and d_3, and J. C. Smith defined the functions d_6 and d_7. The following relations hold for all metric spaces (X, ρ):

$$d_2(X, \rho) \leqq d_3(X, \rho) \leqq d_6(X, \rho) \leqq d_7(X, \rho) \leqq d_0(X, \rho) .$$

Since all of the metric-dependent dimension functions above satisfy a "Weak Sum Theorem," it is natural to ask if any of these functions satisfy the Finite Sum Theorem or the Countable Sum Theorem. In this paper the authors obtain new properties of these dimension functions, and using these results construct examples for which none of the metric dependent dimension functions satisfy either of the sum theorems in question.

Let d_0 denote the metric dimension function defined by Katětov [2], and let dim be the covering dimension function. K. Nagami and J. H. Roberts [5] introduced the metric-dependent dimension functions d_2 and d_3, and J. C. Smith [7] defined the functions d_6 and d_7. The following relations hold for all metric spaces (X, ρ):

$$(*) \qquad d_2(X, \rho) \leqq d_3(X, \rho) \leqq d_6(X, \rho) \leqq d_7(X, \rho) \leqq d_0(X, \rho) .$$

In [8] J. C. Smith has shown that all of the above dimension functions satisfy the "Weak Sum Theorem" stated below for d_2.

THEOREM. *Let (X, ρ) be a metric space satisfying these conditions:*

(1) $X = \bigcup_{\alpha \in A} F_\alpha$, *where each* F_α *is closed in* X.

(2) $\{F_\alpha : \alpha \in A\}$ *is locally finite.*

(3) $d_2(F_\alpha, \rho) \leqq n$ *for each* $\alpha \in A$.

(4) $\dim [(\text{bdry } F_\alpha) \cap F_\beta] \leqq n - 1$ *for* $\alpha \neq \beta$.

Then $d_2(X, \rho) \leqq n$.

It is now natural to ask the following question. Do any of the above dimension functions satisfy the Countable Sum Theorem or the Finite Sum Theorem? In this paper we answer this question in the negative. In § 2 we obtain a number of results relating the dimension functions d_2 and d_0 to certain subsets of Euclidean n-space. In § 3 we apply these results in constructing a metric space for which none of the above metric-dependent dimension functions satisfies the Finite Sum Theorem.[1] In § 4 we prove that if any countable disjoint collection of compact subsets of Euclidean n-space ($n \geqq 3$) is removed, the dimension function d_2 may decrease by at most 1. This result is analogous to Theorem 1 of [5]. As an application of this theorem we construct an example of a metric space for which none of the above metric-dependent dimension functions satisfies the Countable Sum Theorem.

2. Definitions and preliminary results.

DEFINITION 2.1. Let (X, ρ) be a metric space and let B, C and D be closed subsets of X. The set B is said to *separate* C and D in X if $X - B = S \cup T$ where S and T are nonempty open sets, $C \subseteqq S$ and $D \subseteqq T$.

DEFINITION 2.2. Let (X, ρ) be a nonempty metric space and let n be a nonnegative integer. Then $d_2(X, \rho) \leqq n$ if (X, ρ) satisfies this condition:

(D_2) For any collection $\mathscr{C} = \{(C_i, C'_i): i = 1, \cdots, n+1\}$ of $n+1$ pairs of closed sets with $\rho(C_i, C'_i) > 0$ for each $i = 1, \cdots, n+1$, there exist closed sets B_i, $i = 1, \cdots, n+1$, such that

(i) B_i separates C_i and C'_i for each $i = 1, \cdots, n+1$ and

(ii) $\bigcap_{i=1}^{n+1} B_i = \varnothing$.

If $X = \varnothing$ then $d_2(X, \rho) = -1$.

DEFINITION 2.3. Let (X, ρ) be a nonempty metric space. The *metric dimension* of (X, ρ), written $d_0(X, \rho)$, is the smallest integer k such that for every $\varepsilon > 0$ there exists an open cover $\mathscr{U}$ of X with mesh $(\mathscr{U}) < \varepsilon$ and order $(\mathscr{U}) \leqq k + 1$.

DEFINITION 2.4. Let d denote a dimension function on the class of all metric spaces. Then d is said to have the *Finite Sum Property* if given any metric space (X, ρ) which is the finite union of closed subsets A_i with $d(A_i, \rho) \leqq n$, then $d(X, \rho) \leqq n$. Also d is said to have the *Monotone Sum Property* if given any metric space (X, ρ) which is the countable union of closed subspaces A_i with $A_i \subseteqq A_{i+1}$ and $d(A_i, \rho) \leqq n$ for each i, then $d(X, \rho) \leqq n$.

DEFINITION 2.5. Let (X, ρ) be a metric space. Suppose

[1] The basic idea for Example 3.1 is due to J.H. Roberts.

$$\mathscr{C} = \{(C_i, C_i'): i = 1, \cdots, n\}$$

is a collection of n pairs of disjoint closed subsets of X with the property that, if B_i is a closed set separating C_i and C_i' in X for each i, then $\bigcap_{i=1}^{n} B_i \neq \emptyset$. Then $\mathscr{C}$ will be called an *n-defining system for X*.

DEFINITION 2.6. A decomposition of a metric space (X, ρ), $X = \bigcup_{\alpha \in A} H_\alpha$ will be called a *proper decomposition* if there are at least two indices $\alpha, \beta \in A$ such that $H_\alpha \neq \emptyset$ and $H_\beta \neq \emptyset$.

The following characterization of d_0 is proved in [5] page 426.

LEMMA 2.7. *Let (X, ρ) be a metric space. Then $d_0(X, \rho) \leqq n$ if and only if there exists a sequence of locally finite closed coverings $\{\mathscr{F}_i: i \geqq 1\}$ such that*

(i) *mesh $(\mathscr{F}_i) < 1/i$ for each i.*

(ii) *ord $(\mathscr{F}_i) \leqq n + 1$ for each i.*

The following theorem is proved in [4].

THEOREM 2.8. *Let X be a metric space with $\dim X \leqq n$, and let $B_0, B_1, \cdots$ be a sequence of closed subsets of X such that $B_0 = X$ and $\dim (B_i) = n_i$. Let $\varepsilon > 0$. Then there exists a locally finite closed covering $\mathscr{F} = \{F_\alpha: \alpha \in \Gamma\}$ which satisfies the following conditions:*

(i) *mesh $(\mathscr{F}) < (\varepsilon)$.*

(ii) *for each i, ord $(\mathscr{F} \mid B_i) \leqq n_i + 1$.*

(iii) *for each i and for each $j \leqq n_i + 2$, $\dim \bigcap_{k=1}^{j} [F_{\alpha(k)} \cap B_i] \leqq n_i - j + 1$, where $\{\alpha(1), \alpha(2), \cdots, \alpha(j)\}$ is any collection of j distinct members of Γ.*

THEOREM 2.9. *Let X be a metric space with $\dim X \leqq n$, and let $B_0, B_1, \cdots$ be a sequence of closed subsets of X with $\dim (B_i) = n_i$ and $B_0 = X$.* Let $\varepsilon > 0$. *Then there exists a locally finite closed covering $\mathscr{F} = \{F_\alpha: \alpha \in \Gamma\}$ which satisfies the following conditions:*

(i) *mesh $(\mathscr{F}) < \varepsilon$.*

(ii) *if $A_j = \{x: \mathrm{ord}\,(x, \mathscr{F}) \geqq j\}$ for $j = 1, \cdots, n + 1$, then*

$$\dim (A_j \cap B_i) \leqq n_i - j + 1$$

for each i.

Proof. By Theorem 2.8 above there exists a locally finite closed cover $\mathscr{F} = \{F_\alpha: \alpha \in \Gamma\}$ which satisfies (i)–(iii) of that theorem. By (ii) with $B_0 = X$, we have ord $\{\mathscr{F}\} \leqq n + 1$. For each k, $1 \leqq k \leqq n + 1$, define $H_k = \{x: \mathrm{ord}\,(x, \mathscr{F}) = k\}$ so that $A_j = \bigcup_{k=j}^{n+1} H_k$. Let Γ_k

be the collection of all subsets of Γ whose cardinality is k, and let $\mathscr{L}_k = \{\bigcap_{\alpha \in g} F_\alpha : g \in \Gamma_k\}$. Since $\mathscr{F}$ is locally finite, $\mathscr{L}_k$ is a locally finite collection of closed sets whose union is A_k. Thus by (iii) in Theorem 2.8 we have that $\dim (L \cap B_i) \leqq n_i - j + 1$ for each $L \in \mathscr{L}_k$. Hence by the Locally Finite Sum Theorem for covering dimension, $\dim (A_j \cap B_i) \leqq n_i - j + 1$ for each i.

As an application of Theorem 2.9 above we obtain the following.

THEOREM 2.10. *Let $\{J_i : i \geqq 1\}$ be a sequence of closed substs of a Cantor 3-manifold (K, ρ) such that $\dim (J_i) \leqq 1$ for each i. Then there exists a sequence of closed sets $\{H_i : i \geqq 1\}$ satisfying the following properties*:

(1) *For each i, $H_i \subseteqq K - \bigcup_{k \geqq 1} J_k$.*

(2) *$H_i \cap H_j = \varnothing$ for $i \neq j$.*

(3) *$\dim (H_i) \leqq 1$ for each i.*

(4) *$d_0(K - [(\bigcup_{i \geqq 1} J_i) \cup (\bigcup_{i \geqq 1} H_i)]) \leqq 1$.*

Proof. Since (K, ρ) is a Cantor 3-manifold there exists a 3-defining system $\mathscr{C}_3 = \{(C_i, C_i') : i = 1, 2, 3\}$ for (K, ρ). We may assume that $\rho(C_i, C_i') > \delta > 0$. By Theorem 2.9 with $B_i = J_i$ for $i \geqq 1$, there exists a finite closed cover $\mathscr{F}_1$ of K satisfying,

(1) mesh $(\mathscr{F}_1) < \delta$

(2) if $A_j^1 = \{x : \operatorname{ord}(x, \mathscr{F}_1) \geqq j\}$ for $j = 1, \cdots, 4$ then

$$\dim (A_j^1 \cap B_0) \leqq 3 - j + 1 .$$

Let $H_0 = B_0$ and $H_1 = A_3^1$. By (2) above we have $\dim (A_3^1) \leqq 1$. By Theorem 1 in [6] we have $\dim (A_3^1) \geqq 1$. Hence $\dim (A_3^1) = 1$. Therefore by Theorem 2.9, $\dim (H_1 \cap J_i) \leqq 1 - 3 + 1 = -1$, so that $H_1 \cap J_i = \varnothing$ for all $i \geqq 1$.

We apply Theorem 2.9 again with $B_i = H_i$ for $i = 0, 1$ and $B_i = J_{i-1}$ for $i \geqq 2$, and $\varepsilon = \delta/2$. Thus there exists a finite closed cover $\mathscr{F}_2$ of K satisfying.

(1) mesh $(\mathscr{F}_2) < \delta/2$.

(2) if $A_j^2 = \{x : \operatorname{ord}(x, \mathscr{F}_2) \geqq j\}$ for $j = 1, \cdots, 4$ then

$$\dim (A_j^2 \cap H_i) \leqq n_i - j + 1 \qquad \text{for } i = 0, 1 .$$

Let $H_2 = A_3^2$. By the same argument as above we have $\dim (H_2) = 1$ and $\dim (H_1 \cap H_2) \leqq -1$, so that $H_1 \cap H_2 = \varnothing$. Similarly $H_i \cap J_k = \varnothing$ for $i = 1, 2$ and any $j \geqq 1$.

Repeating this process we obtain a sequence of finite closed covers $\{\mathscr{F}_i : i \geqq 1\}$ satisfying the following:

(1) mesh $(\mathscr{F}_i) < \delta/i$ for each $i \geqq 1$.

(2) with $H_i = A_3^i$ for each $i \geqq 1$, we have $\dim (H_i) = 1$, and H_i

is a closed subset of K.

(3) $H_i \cap H_j = \varnothing$ for all $i \neq j$, and $H_i \cap J_k = \varnothing$ for all $i \geqq 1$, $k \geqq 1$.

Let $X = K - [(\bigcup_{i \geqq 1} H_i) \cup (\bigcup_{i \geqq 1} J_i)]$. Then $d_0(X, \rho) \leqq 1$ by Lemma 2.7.

The following Lemma is proved in [3, p. 21]

LEMMA 2.11. *Let X be a compact metric space with disjoint closed subsets D and E. Then one of the following must be true.*

(1) *There exists a continuum W in X such that $W \cap D \neq \varnothing$ and $W \cap E \neq \varnothing$.*

(2) *The sets D and E can be separated by the empty set.*

The following is an immediate consequence of Lemma 2.11.

LEMMA 2.12. *Let (X, ρ) be a compact metric space, let*

$$\mathscr{C} = \{(C_i, C_i'): i = 1, \cdots, n\}$$

be an n-defining system for X, and let $Y \subseteqq X$ such that

$$d_2(X - Y) \leqq n - 2 .$$

If for each $i = 1, \cdots, n-1$, B_i is a closed set separating C_i from C_i', then there exists a continuum $W \subseteqq \bigcap_{i=1}^{n-1} B_i \subseteqq Y$ such that $W \cap C_n \neq \varnothing$ and $W \cap C_n' \neq \varnothing$.

The following is proved in [5, p. 416]

LEMMA 2.13. *If X is a connected compact Hausdorff space then there is no countable proper decomposition of X into mutually disjoint closed subsets.*

The following is an easy consequence of Lemma 2.13.

LEMMA 2.14. *If (X, ρ) is a closed connected subset of Euclidean n-space, then there is no proper decomposition of X into a countable collection of mutually disjoint, compact sets.*

3. No metric-dependent dimension function has the finite sum property.

EXAMPLE 3.1. Using Theorem 2.10 we construct a metric space (X, ρ) with the property that $X = A_1 \cup A_2$, where

$$d_0(A_1, \rho) \leqq 1, d_0(A_2, \rho) \leqq 1$$

but $d_2(X, \rho) \geqq 2$. Hence by the relation (*) above (X, ρ) is an example for which none of the metric-dependent dimension functions have the Finite Sum Property.

Let $Y_1 = \{(x_1, x_2, x_3): 0 \leqq x_i \leqq 1 \quad i = 1, 2, 3\}$. Since (Y_1, ρ) is a Cantor 3-manifold by Theorem 2.10, with $J_i = \varnothing$ for all $i \geqq 1$, there exists a sequence of sets $\{H_i: i \geqq 1\}$ satisfying:

(1) each H_i is closed in Y_1.
(2) $H_i \cap H_j = \varnothing$ for $i \neq j$.
(3) $\dim(H_i) \leqq 1$ for all i.
(4) $d_0(Y_1 - \bigcup_{i \geqq 1} H_i) \leqq 1$.

Similarly let $Y_2 = \{(x_1, x_2, x_3): 0 \leqq x_i \leqq 1$ for $i = 2, 3$ and $1 \leqq x_1 \leqq 2\}$. Again by Theorem 2.10 with $J_i = H_i$ for each $i \geqq 1$, there is a sequence of sets $\{L_i: i \geqq 1\}$ with the following properties:

(5) each L_i is closed in Y_2 and $L_i \subseteqq Y_2 - \bigcup_{k \geqq 1} H_k$.
(6) $L_i \cap L_j = \varnothing$ for $i \neq j$.
(7) $\dim(L_i) \leqq 1$ for all $i \geqq 1$.
(8) $d_0(Y_2 - (\bigcup_{k \geqq 1} H_k \cup \bigcup_{k \geqq 1} L_k)) \leqq 1$.

Let $A_1 = Y_1 - (\bigcup_{i \geqq 1} H_i \cup \bigcup_{i \geqq 1} L_i)$, let $A_2 = Y_2 - (\bigcup_{i \geqq 1} H_i \cup \bigcup_{i \geqq 1} L_i)$ and define $X = A_1 \cup A_2$. Then $d_0(A_1, \rho) \leqq 1$ and $d_0(A_2, \rho) \leqq 1$ by (4) and (8) above. We assert that $d_2(X, \rho) \geqq 2$. Suppose $d_2(X, \rho) \leqq 1$. Let

$$C_1 = \{(x_1, x_2, x_3): x_1 = 0; 0 \leqq x_i \leqq 1, i = 1, 2\}$$

$$C_1' = \{(x_1, x_2, x_3): x_1 = 2; 0 \leqq x_i \leqq 1, i = 1, 2\}$$

and let

$$C_i = \{(x_1, x_2, x_3): 0 \leqq x_1 \leqq 2; x_i = 0; 0 \leqq x_j \leqq 1, j \neq i\}$$

$$C_i' = \{(x_1, x_2, x_3): 0 \leqq x_1 \leqq 2; x_i = 1; 0 \leqq x_j \leqq 1, j \neq i\} \quad \text{for } i = 2, 3 .$$

Then $\mathscr{C} = \{(C_i, C_i'): i = 1, 2, 3\}$ is a 3-defining system for the compact metric space $Z = Y_1 \cup Y_2$. By Lemma 2.12 there exists a continuum $G \subseteqq (\bigcup_{i \geqq 1} H_i) \cup (\bigcup_{i \geqq 1} L_i)$ such that $G \cap C_1 \neq \varnothing$ and $G \cap C_1' \neq \varnothing$. Since $H_i \cap C_1' = \varnothing$ and $L_i \cap C_1 = \varnothing$ for each i, G is not contained in any one H_i or L_i. Define $M_{2i} = G \cap H_i$ and $M_{2i-1} = G \cap L_i$ for each $i \geqq 1$. Then $G = \bigcup_{i \geqq 1} M_i$ and $M_i \cap M_j = \varnothing$ for $i \neq j$. The collection $\{M_i: i \geqq 1\}$ is thus a proper decomposition of G. This contradicts Lemma 2.13 above. Therefore $d_2(X, \rho) \geqq 2$.

4. No metric-dependent dimension function has the monotone sum property.

THEOREM 4.1. *Let (E^n, ρ) denote Euclidean n-space for $n \geqq 3$. Let $\{A_i: i \geqq 1\}$ be any collection of compact subsets of E^n such that*

$A_i \cap A_j = \varnothing$ *for all* $i \neq j$. *Then* $d_2(E^n - \bigcup_{i \geqq 1} A_i) \geqq n - 1$.

Proof. Suppose $d_2(E^n - \bigcup_{i \geqq 1} A_i) \leqq n-2$. For each $i = 1, \cdots, n-1$ let $C_i = \{(x_1, \cdots, x_n); x_i = 1\}$ and let $C'_i = \{(x_1, \cdots, x_n): x_i = -1\}$. For each $j \geqq 1$, define $S_j = \{(x_1, \cdots, x_n): x_n = j\}$ and let

$$S_0 = \{(x_1, \cdots, x_n): x_n = 0, 0 \leqq x_i \leqq 1, i = 1, \cdots, n-1\} .$$

Then for any $j \geqq 1$ the collection

$$\mathscr{C}_j = \{(C_i, C'_i): i = 1, \cdots, n-1\} \cup (S_j, S_0)\}$$

is an n-defining system for the compact space

$$T_j = \{(x_1, \cdots, x_n): |x_i| \leqq j\} .$$

Since $d_2(E^n - \bigcup_{i \geqq 1} A_i) \leqq n-2$, there exist closed sets $B_1, \cdots, B_{n-1}$ such that B_i separates C_i from C'_i for each $i = 1, \cdots, n-1$, and $B = \bigcap_{i=1}^{n-1} B_i \subseteqq \bigcup_{i \geqq 1} A_i$. By Lemma 2.12, for each $j \geqq 1$ there exists a continuum D_j such that $D_j \subseteqq B \subseteqq \bigcup_{i \geqq 1} A_i$, $D_j \cap S_0 \neq \varnothing$ and $D_j \cap S_j \neq \varnothing$. If $0 < k < j$, then S_k separates S_0 from S_j in T. Thus for all $j \geqq 1$ and all k satisfying $0 < k \leqq j$, we have that $D_j \cap S_k \neq \varnothing$.

We have thus proved the statement:

(1) For each $j \geqq 1$, B contains the continuum D_j such that $D_j \cap S_k \neq \varnothing$ for all k, $0 \leqq k \leqq j$.

Since $\liminf \{D_j: j \geqq 1\} \neq \varnothing$ we have by [1, p. 100] that in any T_j, $R = \limsup \{D_j: j \geqq 1\}$ is connected. Note that $R \subseteqq B$ since B is closed. From statement (1) above we now have,

(2) R is a connected set with the property that $R \cap S_j \neq \varnothing$ for every $j \geqq 1$.

Since each A_i is compact, R cannot be contained in any one A_i. Let $H_i = A_i \cap R$. Then $\{H_i: i \geqq 1\}$ is a proper decomposition of the connected set R into a collection of mutually disjoint compact sets. This contradicts Lemma 2.14 and completes the proof of the theorem.

EXAMPLE 4.2. We construct a metric space (X, ρ) with the property that $X = \bigcup_{i \geqq 1} A_i$, where for each $i \geqq 1$, A_i is a closed set, $d_0(A_i, \rho) \leqq 1$ and $A_i \subseteqq A_{i+1}$, yet $d_2(X, \rho) \geqq 2$. For each $i \geqq 1$, let $T_i = \{(x_1, x_2, x_3): |x_i| \leqq i\}$. Then $\bigcup_{i \geqq 1} T_i = E^3$ and $T_i \subseteqq T_{i+1}$ for all $i \geqq 1$. For each $i \geqq 1$ we construct a sequence of closed subsets $\{H_{ik}: k \geqq 1\}$ of T_i. Applying Theorem 2.10 to the Cantor 3-manifold T_1, with $J_k = \varnothing$ for all $k \geqq 1$, we obtain a sequence of closed subsets

$$\{H_{1k}: k \geqq 1\}$$

of T_1 such that:

(1) $H_{1k} \cap H_{1j} = \varnothing$ for all $j \neq k$.

(2) $\dim(H_{1k}) \leqq 1$ for all $k \geqq 1$.

(3) $d_0(T_1 - \bigcup_{k\geqq 1} H_{1k}) \leqq 1$.

Suppose that for each $i = 1, \cdots, m$ the closed collection $\{H_{ik}: k \geqq 1\}$ has been constructed satisfying the following,

(1) $H_{ik} \cap H_{lj} = \varnothing$ for $(i, k) \neq (1, j)$; $1 \leqq i \leqq m$, $1 \leqq l \leqq m$.

(2) $\dim(H_{ik}) \leqq 1$ for all i, $1 \leqq i \leqq m$, and for all $k \geqq 1$.

(3) $d_0(T_i - \bigcup_{j=1}^{i} [\bigcup_{k\geqq 1} H_{jk}]) \leqq 1$.

To construct the collection $\{H_{m+1,k}: k \geqq 1\}$ we apply Theorem 2.10 again to the Cantor 3-manifold T_{m+1}, identifying the collection $\{J_i: i \geqq 1\}$ in the theorem with the collection $\{H_{jk}: j = 1, \cdots, m; k \geqq 1\}$. Finally we conclude that there exists a countable collection of compact sets $\{H_{ik}: i \geqq 1\ k \geqq 1\}$ satisfying the following:

(1) $H_{ik} \cap H_{lj} = \varnothing$ for all $(i, k) \neq (l, j)$

(2) $\dim(H_{ik}) \leqq 1$ for all i, k.

(3) $d_0(T_i - \bigcup_{j=1}^{i} [\bigcup_{k\geqq 1} H_{jk}]) \leqq 1$, for all i.

For each $i \geqq 1$, we define $W_i = T_i - \bigcup_{j=1}^{i} [\bigcup_{k\geqq 1} H_{jk}]$,

$$A_i = W_i - \bigcup_{j=1}^{\infty} \left[\bigcup_{k\geqq 1} H_{jk} \right],$$

and $X = \bigcup_{i\geqq 1} A_i$. Now X is a monotone sum since for all $i \geqq 1$, $A_i \subseteqq A_{i+1}$, and A_i is closed in X. Also we have that

$$X = E^3 - \bigcup_{j=1}^{\infty} \left[\bigcup_{k=1}^{\infty} H_{jk} \right]$$

and for each i, $d_0(W_i) \leqq 1$ by construction. Therefore $d_0(A_i) \leqq 1$ since $A_i \subseteqq W_i$ for each i.

Now the collection $\{H_{jk}: j \geqq 1, k \geqq 1\}$ is a countable collection of compact mutually disjoint subsets of E^3. Hence by Theorem 4.1 above we have that $d_2(X) \geqq 2$. Thus (X, ρ) is an example of a metric space for which none of the metric-dependent dimension functions satisfy the Monotone Sum Property.

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