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**ON COMPLETELY HAUSDORFF-COMPLETION OF A
COMPLETELY HAUSDORFF SPACE**

ASIT BARAN RAHA

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R. M. Stephenson, Jr. (Trans. Amer. Math. Soc. 133 (1968), 537-546) has established the existence of a completely Hausdorff-closed extension X' of an arbitrary completely Hausdorff space X . Stephenson demonstrates that X' enjoys many interesting properties of the Stone-Čech compactification. This paper shows that, by a modification of the topology, X' is made also to possess a property which is in the line of the celebrated property of the Stone-Čech compactification of a completely regular Hausdorff space that it is the largest amongst all Hausdorff compactifications.

1. Introduction. A topological space X is called completely Hausdorff if for every pair x, y of distinct points of X there exists a continuous real valued function f on X such that $f(x) \neq f(y)$. A completely Hausdorff space is called completely Hausdorff-closed if every homeomorphic image of it in any completely Hausdorff space is closed. A space Y is termed a completely Hausdorff-closed extension of a completely Hausdorff space X if X is dense in Y and Y is completely Hausdorff-closed. R. M. Stephenson, Jr. in [4] has established the existence of a completely Hausdorff-closed extension (referred to as the completely Hausdorff-completion) X' of an arbitrary completely Hausdorff space X . If X is completely regular (which, of course, assumes Hausdorff property and is necessarily completely Hausdorff) then X' is the Stone-Čech compactification of X . Stephenson shows {[4], Theorem 4} that, even if X is completely Hausdorff but not necessarily completely regular, X' continues to enjoy many interesting properties of the Stone-Čech compactification. By enlarging the topology of X' we shall, in fact, strengthen Theorem 4 of [4] in the sense that property (vii) therein will be replaced by the following:

X' is a projective maximum in the class of completely Hausdorff-closed extensions Y of X with the property that any element in $F(X)$, the set of all continuous functions on X into $[0, 1]$, admits an extension to $F(Y)$.

The above property is, obviously, akin to the well-known fact that the Stone-Čech compactification is largest among the Hausdorff compactifications of a completely regular Hausdorff space.

2. Notations and definitions. We shall try to follow the notations and definitions of [4] as far as possible.

$C(X)$ will stand for the set of all bounded continuous functions on X . If Z is any topological space, we shall denote by $C(X, Z)$ the set of all continuous mappings of X into Z .

A topological space Y is an *extension space* of another space X if X is dense in Y . If T is an extension space of a topological space S , the *tracefilters* of T are the filters $\mathcal{F}(t)$, $t \in T - S$, where $\mathcal{F}(t)$ is the filter on S given by $\{U \cap S: U \text{ a neighbourhood of } t \text{ in } T\}$.

Banaschewski [1] introduced the notion of a projective maximum in a set E of extensions of X ; an extension Y in E is a projective maximum in E if for each Z in E there is a continuous function from Y onto Z which leaves X pointwise fixed.

A filter \mathcal{F} on a space X is called completely regular provided that it has a base \mathcal{B} of open sets such that for each $B \in \mathcal{B}$, there is a set $B' \subset B$ in \mathcal{B} and a function $f \in F(X)$ such that $f(B') = 0$ and $f = 1$ on $X - B$.

3. **Main result.** Let X be a completely Hausdorff space, and let \mathcal{M} be the set of all maximal completely regular filters on X which have empty adherences. (If \mathcal{F} is a completely regular filter,

$$\cap \{F: F \in \mathcal{F}\} = \cap \{\bar{F}: F \in \mathcal{F}\} = \text{adherence of } \mathcal{F} ,$$

where \bar{F} = closure of F in X . If $\cap \{F: F \in \mathcal{F}\} = \emptyset$, \mathcal{F} is called *free*, otherwise it is called *fixed*.) Put $X' = X \cup \mathcal{M}$. We shall endow X' with a topology as follows:

Any set, open in X , is also open in X' . If $\mathcal{F} \in \mathcal{M}$, basic neighbourhoods of \mathcal{F} are of the form $G \cup \{\mathcal{F}\}$ where $G \in \mathcal{F}$. With this topology (will, henceforth, be called the Katětov topology) X' becomes a completely Hausdorff-closed space and will be called the completely Hausdorff-completion of X . The trace filters of X' are the filters $\{\mathcal{F}(\mathcal{F}): \mathcal{F} \in \mathcal{M}\}$ and for each $\mathcal{F} \in \mathcal{M}$, $\mathcal{F}(\mathcal{F}) = \{U \cap X: U \subset X' \text{ and } U \text{ a neighbourhood of } \mathcal{F}\} = \{G: G \in \mathcal{F}\} = \mathcal{F}$. Thus the trace filters of X' are the maximal completely regular filters \mathcal{F} on X such that

$$\cap \{G: G \in \mathcal{F}\} = \emptyset .$$

Now we are in a position to state our main theorem which is identical with Theorem 4 of [4] with the exception of property (vii).

THEOREM 1. *Let X be a completely Hausdorff space. The completely Hausdorff-completion X' of X has the following properties:*

- (i) *If Z is a compact Hausdorff space, then each function in $C(X, Z)$ has a unique extension in $C(X', Z)$.*
- (ii) *The Stone-Weierstrass theorem holds for X' .*

(iii) X' is locally connected if and only if X is locally connected and each trace filter of X' has a base consisting of connected open sets.

(iv) X' is locally connected only if X is locally connected and pseudocompact.

(v) X' is connected if and only if X is connected.

(vi) $C(X')$ and $C(X)$ are isomorphic, and if R is the real line, $C(X')$ and $C(X, R)$ are isomorphic only if X is pseudocompact.

(vii) Suppose Y is a completely Hausdorff-closed space containing X as a dense subset and each element of $F(X)$ has an extension to $F(Y)$. Then there exists a one-to-one function $g \in C(X', Y)$ such that $g(X') = Y$ and g is identity on X . In short, X' is a projective maximum in the class of completely Hausdorff-closed extensions Y of X with the property that any element in $F(X)$ admits an extension to $F(Y)$.

Proof. Proofs of (i) – (vi) are omitted as they are same as those given in Theorem 4 of [4] (page 540). We shall only give a proof for (vii). Let Y be a completely Hausdorff-closed topological space containing X as a dense subset and such that every function in $F(X)$ admits an (unique) extension to $F(Y)$. If \mathcal{F} is a nonconvergent maximal completely regular filter on X (i.e., $\mathcal{F} \in \mathcal{M}$) define $Z = \{f \in F(X) : \text{for some } G', G \in \mathcal{F} \text{ with } G' \subset G, \text{ one has } f(G') = 0 \text{ and } f(X - G) = 1\}$. Z is nonvoid as \mathcal{F} is completely regular. For $f \in F(X)$ let f' denote its extension in $F(Y)$. Put $Z' = \{f' : f \in Z\}$. Take $\mathcal{S} = \{V(f', t) = f'^{-1}[0, t) : f' \in Z', 0 < t \leq 1\}$. The empty set does not belong to \mathcal{S} . Consider, $V(f'_i, t_i) \in \mathcal{S}, i = 1, 2, \dots, n$ and choose, for each $i, 0 < s_i < t_i$. By using the normality of $[0, 1]$ we can get $g_i \in F(Y)$ such that $g_i(V(f'_i, s_i)) = 0$ and $g_i[Y - V(f'_i, t_i)] = 1$ for $i = 1, 2, \dots, n$. Put $g = \max_{1 \leq i \leq n} g_i$. Then $g \in F(Y)$ and $g[\bigcap_{i=1}^n V(f'_i, s_i)] = 0$ and

$$g[Y - \bigcap_{j=1}^n V(f'_j, t_j)] = 1.$$

Note also that $\bigcap_{j=1}^n V(f'_j, s_j) \subset \bigcap_{j=1}^n V(f'_j, t_j)$. Thus, we have shown that finite intersections of sets of \mathcal{S} form a completely regular filter base on Y . Let \mathcal{G} be the completely regular filter on Y generated by \mathcal{S} and let \mathcal{U} denote a maximal completely regular filter on Y such that $\mathcal{G} \subset \mathcal{U}$. Since Y is completely Hausdorff-closed every completely regular filter on Y has nonempty adherence (See [4] Theorem 1, and [2]). Consequently adherence of \mathcal{U} ($= ad(\mathcal{U})$) is nonempty and maximality of \mathcal{U} will make \mathcal{U} converge to each point in $ad(\mathcal{U})$. But Y is Hausdorff, so $ad(\mathcal{U})$ must contain exactly one point, i.e., $\bigcap U = \bigcap \{U : U \in \mathcal{U}\}$ is a singleton. We now claim that $\mathcal{F} = \{U \cap X : U \in \mathcal{U}\}$.

Proof of the claim. Since \mathcal{U} is a maximal completely regular

open filter it has a completely regular filter base \mathcal{V} consisting of open sets. As X is dense in Y , it is easy to see that $\mathcal{U} \cap X = \{U \cap X : U \in \mathcal{U}\}$ is an open filter on X with an open base given by $\mathcal{V} \cap X = \{V \cap X : V \in \mathcal{V}\}$. Let $V \cap X \in \mathcal{V} \cap X$. Since $V \in \mathcal{V}$ there exist $V' \in \mathcal{V}$ with $V' \subset V$ and $h \in F(Y)$ such that $h(V') = 0$ and $h(Y - V) = 1$. Obviously, $h(V' \cap X) = 0$ and $h(X - V \cap X) = 1$. Let f denote the restriction of h to X . Then $f \in F(X)$ and $f(V' \cap X) = 0$ and $f(X - V \cap X) = 1$ i.e., $\mathcal{V} \cap X$ is a completely regular filter base on X for $\mathcal{U} \cap X$. Therefore $\mathcal{U} \cap X$ is a completely regular filter on X . Again \mathcal{F} is a completely regular filter on X , so $F \in \mathcal{F}$ implies that there exist $F' \in \mathcal{F}$ with $F' \subset F$ and $f \in F(X)$ such that $f(F') = 0$ and $f(X - F) = 1$. This gives $F' \subset f^{-1}[0, 1) \subset F$. Hence $f \in Z$ and $F' \subset f'^{-1}[0, 1) \cap X \subset F$ where $f' \in Z'$. Now, $f'^{-1}[0, 1) \in \mathcal{D} \subset \mathcal{U}$. Thus $X \cap f'^{-1}[0, 1) \in \mathcal{U} \cap X$ and $F \supset X \cap f'^{-1}[0, 1)$ implies $F \in \mathcal{U} \cap X$ (since it is a filter). We get $\mathcal{F} \subset \mathcal{U} \cap X$ and maximality of \mathcal{F} forces $\mathcal{F} = \mathcal{U} \cap X$. Immediately we have from the above fact, $(\cap U) \cap X = \cap (U \cap X) = \cap \{F : F \in \mathcal{F}\} = \emptyset$ as \mathcal{F} is a free maximal completely regular filter. So the single point contained in $\cap U$ is actually in $Y - X$. Let the point be denoted by $y(\mathcal{F})$. Next we show that if \mathcal{F}_1 and \mathcal{F}_2 are two distinct points in \mathcal{M} , the points $y(\mathcal{F}_1)$ and $y(\mathcal{F}_2)$ are distinct points of $Y - X$. Since \mathcal{F}_1 and \mathcal{F}_2 are two distinct free maximal completely regular filters there must exist $G_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{F}_2$ such that G_1 and G_2 are open in X and $G_1 \cap G_2 = \emptyset$. As shown earlier, we can associate two maximal completely regular filters \mathcal{U}_1 and \mathcal{U}_2 on Y with \mathcal{F}_1 and \mathcal{F}_2 respectively. By definition $\{y(\mathcal{F}_i)\} = \cap \{U : U \in \mathcal{U}_i\}$, $i = 1, 2$ and we also know that $\mathcal{F}_i = \mathcal{U}_i \cap X$. Consequently there exists $U_i \in \mathcal{U}_i$ such that $U_i \cap X = G_i$ and U_i is open ($i = 1, 2$). Since $G_1 \cap G_2 = \emptyset$ and X is dense in Y we have $U_1 \cap U_2 = \emptyset$. Since $y(\mathcal{F}_i) \in U_i$ for $i = 1, 2$ we get $y(\mathcal{F}_1) \neq y(\mathcal{F}_2)$. So far we have shown that $\mathcal{F} \mapsto y(\mathcal{F})$ is a one-to-one map of \mathcal{M} into $Y - X$. Let i denote the identity map on X into Y . Define $\bar{i} : X' \rightarrow Y$ as follows:

$$\begin{aligned} \bar{i}(x) &= i(x) = x && \text{if } x \in X, \text{ and} \\ \bar{i}(\mathcal{F}) &= y(\mathcal{F}) && \text{if } \mathcal{F} \in \mathcal{M} = X' - X. \end{aligned}$$

Claim: \bar{i} is continuous.

We shall establish the continuity by showing the continuity at each point.

(i) Suppose $x \in X$. Then $\bar{i}(x) = x$. Let W be an open neighbourhood of x in Y , then $\bar{i}^{-1}(W) \cap X = i^{-1}(W) = G$, an open neighbourhood of x in X and hence open in X' and also $\bar{i}^{-1}(G) \subset W$.

(ii) For $\mathcal{F} \in \mathcal{M}$, we have $\bar{i}(\mathcal{F}) = y(\mathcal{F})$. By construction of

$y(\mathcal{F})$ we know that it is the point of convergence of a maximal completely regular filter \mathcal{U} on Y such that $\mathcal{F} = \mathcal{U} \cap X$.

If W is an open neighbourhood of $y(\mathcal{F})$ in Y then $W \in \mathcal{U}$ i.e., $W \cap X \in \mathcal{F}$. But $W \cap X$ is open in X and hence $(W \cap X) \cup \{\mathcal{F}\}$ is an open neighbourhood of \mathcal{F} in X' such that

$$\begin{aligned} \bar{i}[(W \cap X) \cup \{\mathcal{F}\}] &= \bar{i}(W \cap X) \cup \bar{i}(\mathcal{F}) = i(W \cap X) \cup \{y(\mathcal{F})\} \\ &= (W \cap X) \cup \{y(\mathcal{F})\} \subset W. \end{aligned}$$

Thus the continuity of \bar{i} has been proved. But X' is, in particular, completely Hausdorff-closed and \bar{i} is a continuous function on X' into a completely Hausdorff space Y in which X is dense. Consequently, from the following fact it will follow that \bar{i} is onto Y .

Fact. Let X be a completely Hausdorff-closed space and let Y be a completely Hausdorff space such that there is a continuous function $f: X \rightarrow Y$. Then $f(X)$ is a completely Hausdorff closed subspace of Y .

Let us put $g = \bar{i}$. Then $g \in C(X', Y)$ with $g(X') = Y$ and g restricted to X equals i , the identity map on X .

COROLLARY 1. *Suppose Y is completely Hausdorff-closed space satisfying the conditions stated in Theorem 1(vii) and f is a homeomorphism of X onto X , then there exists a one-to-one function $g \in C(X', Y)$ such that $g(X') = Y$ and g restricted to X equals f .*

Proof. We first note that if \mathcal{F} is a nonconvergent maximal completely regular filter on X , $f(\mathcal{F})$ is a nonconvergent maximal completely regular filter on X . Then the proof follows by a reasoning similar to one presented in the proof of Theorem 1(vii) where i is replaced by f .

4. REMARKS. The completely Hausdorff-completion X' of X in Theorem 1 is essentially unique, i.e., if T is any completely Hausdorff closed extension of X and T satisfies the properties of Theorem 1 then X' and T are homeomorphic. For there exists $g \in C(X', T)$ such that $g(X') = T$ and g is identity on X . Also, there exists $h \in C(T, X')$ such that $h(T) = X'$ and h is identity on X . Therefore by the following result {[3], page 5} we can assert that X' and T are homeomorphic.

Result. Let X be dense in each of the Hausdorff spaces S and T . If the identity mapping on X has continuous extensions s from

S into T , and t from T into S , then s is a homeomorphism onto, and $s^{-1} = t$.

One can raise the following two questions regarding Theorem 1: (a) Is a Y satisfying the condition (vii) of Theorem 1 homeomorphic to X' ? (b) Is X' a one-to-one continuous image of such Y ? We shall answer both the questions in the negative. Let N denote the set of natural numbers with discrete topology. On N any free maximal completely regular filter is nothing but a free ultrafilter. Thus $\beta N = NU\mathcal{M}$ where \mathcal{M} is the set of all free ultrafilters on N . The topology by which βN is the Stone-Ćech compactification of N will be called Stone-Ćech topology ($S - \check{C}$ topology) for βN . Its open sets are generated by $\{V' : V \text{ open in } N\}$ where $V' = V \cup \{\mathcal{F} \in \mathcal{M} : V \in \mathcal{F}\}$. But, according to our definition, βN endowed with the Katětov topology is the completely Hausdorff-completion of N and in this topology $\mathcal{M} = \beta N - N$ is a closed, discrete infinite subspace of βN and, thus, cannot be compact. While in the $S - \check{C}$ topology of βN , \mathcal{M} is closed, no doubt, and hence compact. Clearly, the $S - \check{C}$ topology is strictly weaker than the Katětov topology. As $S - \check{C}$ topology of βN is compact, *no continuous map from βN with $S - \check{C}$ topology onto βN with Katětov topology can exist*. So homeomorphism is ruled out. But the Stone-Ćech compactification βN satisfies all the conditions enjoyed by a Y in Theorem 1(vii).

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