Pacific Journal of Mathematics

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Vol. 38, No. 1 March 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 1, 1971

BLOCKS AND F-CLASS ALGEBRAS OF FINITE GROUPS

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For an arbitrary field F of characteristic $p \geqq 0$, the usual partitioning of the p-regular elements of a finite group G into F-classes (F-conjugacy classes) is extended to all of G in such a way that the F-classes form a basis of a subalgebra Y of the class algebra Z of G over F. The primitive idempotents of $E \otimes_F Y$, where E is an algebraic closure of F, are the same as those of Z. By means of this fact it is shown that if $p > 0$ the number of blocks of G over F with a given defect group D is not greater than the number of p-regular F-classes L of G with defect group D such that the F-class sum of L in Z is not nilpotent; equality holds if $O_{p,p',p}(G) = G$ or if D is Sylow in G. The results are generalized to arbitrary twisted group algebras of G over F.

1. Introduction. The representation theory of a finite group G over an arbitrary field F involves certain subsets of G called F-conjugacy classes or simply F-classes [6, p. 164], [9, p. 306]. In this paper we show (Theorem 4) that the F-class sums in the group algebra A of G over F form a basis of a subalgebra $Y(A)$ of the center $Z(A)$ of A; we may call $Y(A)$ the F-class algebra of G. (If F has prime characteristic p, the definition of the p-singular F-classes requires some care.) The crucial property of $Y(A)$, from our standpoint, is that its extension $Y(A)^E$ to an algebra over an algebraic closure E of F has precisely the same primitive idempotents as the F-algebra $Z(A)$ (Theorem 4); thus the blocks of G over F correspond to the primitive idempotents of an algebra over an algebraically closed field. Furthermore we obtain a corresponding result for any twisted group algebra (without any normalization of the factor set) of G over F by the methods of [16].

We make use of F-class algebras in conjunction with methods of Berman and Bovdi (Bódi) [2], [3] to obtain results about the number of blocks of twisted group algebras. In the group-algebra case these results (Theorems 6, 8, and 9) can be summarized as follows.

THEOREM 1. *Let F have prime characteristic p. For any p-subgroup D of G, the number of blocks of G over F with D as a defect group is less than or equal to the number of p-regular F-classes L of G with D as a defect group such that the F-class sum of L is not a nilpotent element of A. Equality holds here if* $O_{p,p',p}(G) = G$

or if D is a p-Sylow subgroup of G; in the latter case the nonnilpotence condition can be omitted.

Theorem 1 incorporates generalizations of results of Brauer and Nesbitt [4, Corollaries 1 and 2], [5, (6 D)] as well as of [2] and [3] concerning the case where F is a splitting field for G. In [2, Theorem 2] part of the result for $O_{p,p',p}(G) = G$ is stated for arbitrary F, but without proof. The p-Sylow, or "highest defect", result for group algebras over arbitrary F has been obtained independently by Hubbart [10]; Bovdi's proof of this result is of interest even in the splitting-field case. Treatments of Brauer's results by Rosenberg [17] and Conlon [8] will be referred to frequently. Further references are given below.

In Corollary 2 we generalize a result of Brauer [5, (13A)] on blocks of defect 0. We remark that there is a connection between F-class algebras and the notion of S-rings (see [18] for example).

Added in proof. L. G. Kovács discovered most of Theorem 1 using vertices and sources, but his proof has appeared only in some unpublished notes written by Andrew Hopkins [9a]. Michler [11a] has independently obtained some interesting related results.

Terminology. We have attempted to help a reader interested only in the group-algebra case to skip over the complications caused by twisting. Standard notations, such as $N_G(H)$, $O_{p'}(G)$, $Z(G)$, and the vertical line symbol for restrictions of mappings will be used without comment. A p'-group is one of order not divisible by p, i.e. such that all its elements are p-regular; if $p = 0$, every finite group is a p'-group, and a p-group has order 1. The center and Jacobson radical of an algebra X are called $Z(X)$ and $J(X)$ respectively. We shall follow the notation of [16] except for its categorical machinery.

2. **Representations of a Galois group.** Throughout the paper A denotes a twisted group algebra of a finite group G over an arbitrary field F of characteristic $p \geqq 0$; thus A has a basis $\{a_g : g \in G\}$ with

$$a_g a_{g'} = f(g, g') a_{gg'} , \qquad g, g' \in G , \tag{2.1}$$

for some nonzero $f(g, g') \in F$. For any subset H of G, A_H denotes the subspace of A with basis $\{a_h : h \in H\}$; if H is a subgroup, A_H is a twisted group algebra of H. E is a fixed algebraic closure of F, and $\mathscr{G}$ is the (untopologized) Galois group of E over F. For any F-space (F-algebra) X, $X^E = E \otimes_F X$ is the E-space (E-algebra) obtained from X by extension of the ground field. We regard X as embedded in X^E in the usual way; thus $(A_H)^E = (A^E)_H = A_H^E$.

We consider two representations of $\mathscr{G}$ on the E-space A^E. First there is the well-known canonical semilinear representation of $\mathscr{G}$ on A^E, which we shall call $\boldsymbol{P}_A$: for each $\sigma \in \mathscr{G}$,

$$\left[\sum_{g \in G} w(g) a_g\right] \boldsymbol{P}_A(\sigma) = \sum_{g \in G} w(g)^\sigma a_g \,, \qquad w(g) \in E,$$

where $w(g)^\sigma$ denotes the image of $w(g)$ under σ. $\boldsymbol{P}_A(\sigma)$ is a ring-automorphism of A^E. (The existence of $\boldsymbol{P}_A$ does not depend on the fact that A is a twisted group algebra.)

The second representation of $\mathscr{G}$ on A^E is the linear representation $\boldsymbol{S}_A$ of [16, Theorem 5]. We can describe $\boldsymbol{S}_A(\sigma)$ by the following restatement of [16, Corollary to Theorem 4].

THEOREM 2. *For each $\sigma \in \mathscr{G}$, there is a unique E-linear transformation $\boldsymbol{S}_A(\sigma)$ of A^E to A^E such that:*

(2.2) *For each cyclic subgroup $\langle g \rangle$ of G, the restriction of $\boldsymbol{S}_A(\sigma)$ to $A^E_{\langle g \rangle}$ is an algebra-automorphism of $A^E_{\langle g \rangle}$.*

(2.3) *For each cyclic p'-subgroup $\langle g \rangle$ of G, $\psi_j(a\boldsymbol{S}_A(\sigma)) = [\psi_j(a\boldsymbol{P}_A(\sigma))]^{\sigma^{-1}}$ whenever $a \in A^E_{\langle g \rangle}$ and ψ_j is an irreducible character of $A^E_{\langle g \rangle}$.*

(2.4) *For each cyclic p-subgroup $\langle g \rangle$ of G, $\boldsymbol{S}_A(\sigma)$ fixes every element of $A^E_{\langle g \rangle}$.*

Here ψ_j is defined with values in E. By Theorem 2, the analogue of $\boldsymbol{S}_A(\sigma)$ for any subgroup H of G is

$$\boldsymbol{S}_{A_H}(\sigma) = \boldsymbol{S}_A(\sigma) \,|\, A_H^E \tag{2.5}$$

(cf. [16, Theorem 4, (a)]). The group $\{\boldsymbol{S}_A(\sigma) : \sigma \in \mathscr{G}\}$ is finite [16, § 6].

More explicitly: choose any n divisible by the exponent of G and write $n = n_p n_{p'}$ where n_p is a power of p and $n_{p'}$ is not divisible by p. (If $p = 0$, $n = n_{p'}$.) Choose $m(\sigma)$ so that $\omega^\sigma = \omega^{m(\sigma)}$ for every $n_{p'}$-th root ω of 1 in E and $m(\sigma) \equiv 1 \pmod{n_p}$. Then $\mathscr{G}$ has a permutation representation $\boldsymbol{s}_G$ on G such that

$$g\boldsymbol{s}_G(\sigma) = g^{m(\sigma^{-1})} \,, \qquad g \in G \,. \tag{2.6}$$

Then $a_g \boldsymbol{S}_A(\sigma)$ is a scalar multiple of $a_{g'}$ in A^E where $g' = g\boldsymbol{s}_G(\sigma)$ ([16, (6.4)] gives a formula for the scalar); thus $\boldsymbol{S}_A$ acts monomially, with $\boldsymbol{s}_G$ as the associated permutation representation (cf. [16, § 3]). In particular if A is a group algebra, we can take $a_g = g$; then $g\boldsymbol{S}_A(\sigma) = g\boldsymbol{s}_G(\sigma)$ [16, (9.2)].

G acts by conjugation both on itself and on A^E by automorphisms:

(2.7) $$aK_A(x) = a_x^{-1}aa_x,\ gk_G(x) = x^{-1}gx\ ,$$

for $a \in A^E$, $g \in G$, $x \in G$ [16, (4.1) and (4.2)]; K_A is a monomial representation of G with k_G associated to it. The fixed-point space of K_A is clearly the center $Z(A^E) = Z(A)^E$ of A^E.

In the next proof, and throughout the paper, we shall make tacit use of the basic properties of idempotents of commutative algebras (for example, see [11], especially pp. 54–55). We refer to the primitive idempotents of a commutative algebra as *block idempotents*.

THEOREM 3. *If* $\sigma \in \mathscr{G}$, *then:*

(2.8) $S_A(\sigma) \mid Z(A^E)$ *is an algebra-automorphism of* $Z(A^E)$.

(2.9) *For every block idempotent* d *of* $Z(A^E)$, $dS_A(\sigma) = dP_A(\sigma)$.

Proof. By [16, (8.1)], $S_A(\sigma)K_A(x) = K_A(x)S_A(\sigma)$; this is obvious in the group-algebra case. Hence $S_A(\sigma)$ maps $Z(A^E)$ onto itself. Observe that since $P_A(\sigma)$ permutes the block idempotents, (2.9) says that $S_A(\sigma)$ permutes them in the same way. We prove this theorem in three cases of increasing generality.

Case I. Suppose that A is a group algebra. If also $p = 0$, the theorem is due to Burnside [7, p. 317, Theorem VII]; our argument generalizes his. To each block idempotent d of $Z(A^E)$ there corresponds a "block" $B[d]$ of A^E to which are assigned certain irreducible representations F_j of A^E, their traces or characters φ_j, and the corresponding principal indecomposable representations U_j. Then

(2.10) $$d = \sum_g \sum_j \frac{\deg U_j}{|G|} \varphi_j(g^{-1})g$$

where g runs over the p-regular elements of G and φ_j over the irreducible characters of $B[d]$: this is Osima's formula [12, § 2] written in characteristic p; for $p > 0$ we interpret $(\deg U_j)/|G|$, which can be written with denominator not divisible by p [5, (3F)], as an element of the prime subfield of F. A consideration of characteristic roots shows that $\varphi_j(g^{m(\sigma)}) = \varphi_j(g)^\sigma$ (cf. [16, Theorem 3]) and (2.9) follows. If $p = 0$, $Z(A^E)$ is the direct sum of the fields dE; since $S_A(\sigma) \mid Z(A^E)$ permutes the d's, (2.8) holds. In particular this is true when $F = \boldsymbol{Q}$, in which case any integer relatively prime to the exponent of G can serve as $m(\sigma)$; an easy reduction modulo p yields (2.8) for prime characteristic.

Case II. Suppose that there is a positive integer l such that

$f(g, g')^l = 1$ for all g, g' in (2.1). Then there exists a finite central extension G^* of G such that A is (up to isomorphism) a direct summand of the group algebra A^* of G^* over F [8, pp. 155–156]; then $A = A^*e^*$ for an idempotent e^* of $Z(A^*)$. Let $M: a^* \mapsto a^*e^*$ be the projection of A^* onto A, and let M^E be its extension to a projection of $(A^*)^E$ onto A^E. For any $\sigma \in \mathscr{G}$, set

$$S = \boldsymbol{S}_A(\sigma), S^* = \boldsymbol{S}_{A^*}(\sigma), P = \boldsymbol{P}_A(\sigma), P^* = \boldsymbol{P}_{A^*}(\sigma) .$$

By [16, Theorem 4, (a)], $S^*M^E = M^ES$. Using Case I we find that $e^*S^* = e^*P^* = e^*$, and that for any $z \in Z(A^E)$,

$$zS = (xM^E)S = (zS^*)M^E = (zS^*)e^* = (zS^*)(e^*S^*) = (ze^*)S^* = zS^* ;$$

hence $S|Z(A^E)$ is a restriction of $S^*|Z((A^*)^E)$ and (2.8) holds. As for (2.9), if d is any block idempotent of $Z(A^E)$, $dS = dS^* = dP^* = dP$, using Case I and the fact that $P = P^*|A^E$ by canonicity.

Case III. Let A be arbitrary. By [16, § 9] there exist elements $c(g)$ of E such that if we set $a_g^\# = c(g)a_g$, then $\{a_g^\#: g \in G\}$ is an F-basis of a twisted group algebra $A^\#$ for G over F such that Case II holds for $A^\#$. We have $(A^\#)^E = A^E$. For a fixed $\sigma \in \mathscr{G}$, set $S = \boldsymbol{S}_A(\sigma)$, $S^\# = \boldsymbol{S}_{A^\#}(\sigma)$, $P = \boldsymbol{P}_A(\sigma)$, $P^\# = \boldsymbol{P}_{A^\#}(\sigma)$. At once $P = P^\#T$ where T is the E-linear transformation of A^E onto A^E such that

$$a_g T = \frac{c(g)^\sigma}{c(g)} a_g , \qquad g \in G . \tag{2.11}$$

By the proof of [16, (9.3)], the mapping $g \mapsto c(g)^\sigma/c(g)$ is a 1-cocycle, i.e., a homomorphism of G into the group of roots of unity of E; hence T is an algebra-automorphism.

We claim that $S = S^\#T$. In proving this we can replace G by its cyclic subgroups $\langle g \rangle$ by (2.5). By (2.2) we can suppose that $\langle g \rangle$ is either a p-group or a p'-group. In the first case S and $S^\#$ are the identity by (2.4), and so is T since the homomorphism in (2.11) is trivial. Suppose then that G is a cyclic p'-group. Then $A^E = Z(A^E)$ (see the proof of [16, Theorem 4]) and A^E is the direct sum of the fields dE [8, p. 156]. By (2.3) $\psi_j(dS) = [\psi_j(dP)]^{\sigma^{-1}} = \psi_j(dP)$ for each j since $\psi_j(dP)$ is 0 or 1; hence $dS = dP$ in this case. Similarly $dS^\# = dP^\#$, and $[d(S^\#)^{-1}]S = [d(P^\#)^{-1}]P = dT$; then $S = S^\#T$ for cyclic p'-groups and hence for all G.

Now Case II implies the general case: for since $S^\#|Z(A^E)$ and $T|Z(A^E)$ are algebra-automorphisms, so is $S|Z(A^E)$, while $dS = (dS^\#)T = (dP^\#)T = dP$.

REMARK 1. The argument in Case III shows that (2.3) is equivalent to the condition:

(2.12) *For each cyclic p'-subgroup $\langle g \rangle$ of G, $d\boldsymbol{S}_A(\sigma) = d\boldsymbol{P}_A(\sigma)$ for every block idempotent d of $A^E_{\langle g \rangle}$.*

Hence in Theorem 4 of [16], we can replace condition (b) by our condition (2.9), which is roughly dual to (b). Also condition (c) can be replaced by our stronger condition (2.8).

REMARK 2. Theorem 3 can also be proved using the generalization of (2.10) for twisted group algebras; without proof we state that this formula is

$$d = \sum_g \sum_j \frac{\deg \boldsymbol{U}_j}{|G|} \varphi_j(a_g^{-1})a_g \tag{2.13}$$

with summations as in (2.9). Since $d \in Z(A^E)$, the coefficient of a_g vanishes unless g is in a $\boldsymbol{K}_A$-regular conjugacy class of G (see § 3). Passman [13] has shown that only p-regular g are needed without deriving (2.13).

3. F-class algebras. As in [16, § 8], we can combine $\boldsymbol{S}_A$ and $\boldsymbol{K}_A$ to form a monomial representation $\boldsymbol{D}_A$ of the abstract direct product $\mathscr{G} \times G$ on A^E by setting

$$\boldsymbol{D}_A(\sigma, x) = \boldsymbol{S}_A(\sigma)\boldsymbol{K}_A(x) = \boldsymbol{K}_A(x)\boldsymbol{S}_A(\sigma) , \tag{3.1}$$

$$\boldsymbol{d}_G(\sigma, x) = \boldsymbol{s}_G(\sigma)\boldsymbol{k}_G(x) = \boldsymbol{k}_G(x)\boldsymbol{s}_G(\sigma) . \tag{3.2}$$

The following result was suggested by a lemma of Berman [1, Lemma 3.1].

THEOREM 4. *The fixed-point space of $\boldsymbol{D}_A$ is an E-subalgebra of $Z(A^E)$ with identity. Its block idempotents are identical with those of $Z(A)$.*

Proof. Temporarily denote this space by X. The first sentence follows from (2.8), for since $Z(A^E)$ is the fixed-point space of $\boldsymbol{K}_A$, X is the fixed-point space of the subrepresentation of $\boldsymbol{S}_A$ on $Z(A^E)$. There is a finite normal (not necessarily separable) extension field N of E such that every block idempotent d of $Z(A^E)$ lies in $N \otimes_F Z(A)$. $\boldsymbol{P}_A$ permutes the d's, and by [15, Lemma 2] the block idempotents of $Z(A)$ are the sums $\sum d$ over the various orbits. By (2.9) these are also orbits under $\boldsymbol{S}_A$; then the sums $\sum d$ are the block idempotents of X.

We shall call the orbits of $\boldsymbol{d}_G$ the *F-conjugacy classes*, or *F-classes*, of G. Since $g\boldsymbol{d}_G(\sigma, x) = x^{-1}g^{m(\sigma^{-1})}x$ by (2.6), this agrees with the usual definition [9, p. 306], [1] for the p-regular elements of G (cf. the proof of [16, Theorem 6]). The monomial representation $\boldsymbol{D}_A$ distinguishes certain F-classes: as in [16, § 3] we say that an F-class L is *$\boldsymbol{D}_A$-regular* provided that there exist nonzero $q(g) \in E$, $g \in L$, such that $\boldsymbol{D}_A$ acts as a permutation representation on the elements $q(g)a_g$ of A^E. By [16, Lemma 2] if $g \in L$, then L is $\boldsymbol{D}_A$-regular if and only if the stabilizer $\{(\sigma, x) \in \mathscr{G} \times G: a_g\boldsymbol{D}_A(\sigma, x) = a_g\}$ of a_g under $\boldsymbol{D}_A$ equals the stabilizer of g under $\boldsymbol{d}_G$. (In the group-algebra case, all F-classes are $\boldsymbol{D}_A$-regular.) By [16, Lemma 1] the dimension of the fixed-point space of $\boldsymbol{D}_A$ is the number of $\boldsymbol{D}_A$-regular F-classes. In fact an E-basis is formed by the elements

$$y_L = \sum_{g \in L} q(g)a_g \tag{3.3}$$

as L ranges over the $\boldsymbol{D}_A$-regular F-classes.

Analogous considerations apply to $\boldsymbol{K}_A$: thus we have elements z_K as K ranges over the *$\boldsymbol{K}_A$-regular conjugacy classes* of G which form a well-known basis of the fixed-point space $Z(A^E)$ as well as of $Z(A)$ [8, p. 155].

In the group-algebra case we can choose all $q(g) = 1$ in (3.3) so that the y_L are the *F-class sums* in A. For general A it is interesting, although not essential for our later arguments, that we can choose all $q(g)$ in the ground field F, so that still $y_L \in A$. This statement is equivalent to the following theorem.

THEOREM 5. *The fixed-point space X of $\boldsymbol{D}_A$ has the form $Y(A)^E$ for a unique F-subalgebra $Y(A)$ of $Z(A)$.*

Proof. It will suffice to show that the fixed-point space of $\boldsymbol{S}_A$ has form W^E for an F-subspace W of A, since this will imply that $X = W^E \cap Z(A^E) = [W \cap Z(A)]^E$. By (2.5), (2.2), and (2.5) we can reduce to the case that G is a cyclic p'-group. As in Case III of Theorem 3, $A^E = \bigoplus dE$ and the fixed-point space of $\boldsymbol{S}_A$ is X. By Theorem 4 the block idempotents e of X are all in A; then $X = \bigoplus eE = (\bigoplus eF)^E$ as required. For general G, $Y(A)$ is unique since $Y(A) = X \cap A = X \cap Z(A)$. The statement about the y_L is true since $X = \bigoplus_L [Y(A)^E \cap A_L^E] = \bigoplus_L [Y(A) \cap A_L]^E$.

Henceforth the symbol $Y(A)$ always denotes this F-algebra, and the y_L are chosen in A, so that they form an F-basis of it. $Y(A)$ may be called the *F-class algebra* of A. We could "normalize" the basis $\{a_g\}$ of A, changing it so that all $q(g) = 1$ in (3.3); however we shall not do this in order to avoid conflicting normalizations for subgroups and for conjugacy classes.

We say that an F-class L is *A-nonnilpotent* provided that (a) L

is $\boldsymbol{D}_A$-regular and (b) y_L is not a nilpotent element of $Y(A)$. Here (b) makes sense since y_L is determined up to a scalar multiple; in terms of radicals it is equivalent to saying that $y_L \notin J(Y(A))$ or that $y_L \notin J(Y(A)^E)$. (It is not always true that $J(Y(A)^E) = J(Y(A))^E$: see the example of [15, pp. 12–13].)

REMARK 3. I have not been able to answer the following question even in the group-algebra case: does $\boldsymbol{S}_A(\sigma)$ map $J(A)$ into itself?

4. Counting blocks. From now on p will always be prime. For each F-class L, call any p-Sylow subgroup of $C_G(g)$ for any $g \in L$ a *defect group* of L; this is determined up to conjugacy in G since $C_G(g^{m(\sigma)}) = C_G(g)$. In other words, the defect groups of L are the same as the defect groups of the conjugacy classes within L. Each block idempotent e of $Z(A)$, i.e., of $Y(A)^E$ or of $Y(A)$, has form $e = \sum r[L]y_L$, $r[L] \in F$, summed over the p-regular $\boldsymbol{D}_A$-regular F-classes L (cf. Remark 2). By [17, § 2] and [8, § 3], the largest of the defect groups of those L for which $r[L] \neq 0$ form a single conjugacy class of subgroups of G, called the *defect groups* of e (in A).

The following result is a generalization of the lemma of Brauer that is quoted in its proof.

LEMMA 1. *Let D be any p-subgroup of G, and let $H = N_G(D)$. Then there is a bijection of the set of all $\boldsymbol{D}_A$-regular F-classes of G with defect group D and the set of all $\boldsymbol{D}_{A_H}$-regular F-classes of H with (unique) defect group D, given by $L \mapsto L \cap H$.*

Proof. By a lemma of Brauer [5, (10A)], [17, Lemma 3.4], there is a bijection $K \mapsto K \cap H$ of all conjugate classes of G with defect group D to all conjugate classes of H with unique defect group D. For each F-class L of G with defect group D, $L = \bigcup_{\sigma \in \mathscr{G}} K^{[m(\sigma)]}$ where $K^{[m(\sigma)]} = \{g^{m(\sigma)}: g \in K\}$, and $L \cap H = \bigcup (K \cap H)^{[m(\sigma)]}$; hence there is a bijection $L \mapsto L \cap H$ of all F-classes of G with defect group D to all F-classes of H with defect group D. If L is $\boldsymbol{D}_A$-regular and $h \in L \cap H$, the stabilizers of a_h under $\boldsymbol{D}_A$ and of h under $\boldsymbol{d}_G$ are equal; then the stabilizers of a_h under $\boldsymbol{D}_{A_H}$ and of h under $\boldsymbol{d}_H$ are equal, so that $L \cap H$ is $\boldsymbol{D}_{A_H}$-regular.

Conversely suppose that $L \cap H$ is $\boldsymbol{D}_{A_H}$-regular with defect group D. The following argument is a refinement of the proof of the Lemma of [14]. Let $h \in K \cap H \subseteq L \cap H$, and suppose that $(\sigma, x) \in \mathscr{G} \times G$ is such that $h\boldsymbol{d}_G(\sigma, x) = h$; we must show that $a_h\boldsymbol{D}_A(\sigma, x) = a_h$. Let $T = \{t \in G: a_h\boldsymbol{K}_A(t) = a_h\}$ be the stabilizer of a_h under $\boldsymbol{K}_A$. $K \cap H$ is $\boldsymbol{K}_{A_H}$-regular, i.e., $T \cap H = C_H(h)$. By Brauer's lemma, D is p-Sylow in $C_G(h)$ as well as in $C_H(h)$. Since $C_H(h) \subseteq T \subseteq C_G(h)$, D is p-Sylow in T. Now $a_h\boldsymbol{D}_A(\sigma, x) = ca_h$ for some $c \in E$; if $t \in T$ then

$$\begin{aligned} a_h\boldsymbol{K}_A(x^{-1}tx) &= c^{-1}\, a_h\boldsymbol{D}_A(\sigma, x)\, \boldsymbol{K}_A(x^{-1}\, tx) \\ &= c^{-1}\, a_h\boldsymbol{K}_A(t)\, \boldsymbol{D}_A(\sigma, x) = c^{-1}\, a_h\, \boldsymbol{D}_A(\sigma, x) = a_h\ , \end{aligned}$$

so that $x^{-1}Tx \subseteqq T$; similarly $xTx^{-1} \subseteqq T$, so that $x^{-1}Dx$ is p-Sylow in T. Then $x^{-1}Dx = t^{-1}Dt$ for some $t \in T$, and $xt^{-1} \in N_G(D) = H$. Now

$$h\boldsymbol{d}_H(\sigma, xt^{-1}) = h\boldsymbol{d}_G(\sigma, xt^{-1}) = h\boldsymbol{d}_G(\sigma, x)\boldsymbol{k}_G(t)^{-1} = h\boldsymbol{k}_G(t)^{-1} = h\ .$$

Since $L \cap H$ is $\boldsymbol{D}_{A_H}$-regular, $a_h\boldsymbol{D}_A(\sigma, xt^{-1}) = a_h$; and then $a_h\boldsymbol{D}_A(\sigma, x) = a_h\boldsymbol{K}_A(t) = a_h$ as required.

LEMMA 2 (cf. [3, Lemma 4]). *Under the assumptions of Lemma 1, the number of p-regular A-nonnilpotent F-classes of G with defect group D is not less than the number of p-regular A_H-nonnilpotent F-classes of H with defect group D.*

Proof. The mapping R of A^E into A_H^E defined by

$$\left[\sum_{g \in G} w(g)a_g\right] R = \sum_{g \in C} w(g)a_g\ ,$$

where $C = C_G(D)$, satisfies $\boldsymbol{S}_A(\sigma)R = R\boldsymbol{S}_{A_H}(\sigma)$; hence the Brauer homomorphism $R|Z(A^E)$ of $Z(A^E)$ into $Z(A_H^E)$ [5, (7B)], [17, Lemma 3.3], [8, § 3] carries $Y(A)$ into $Y(A_H)$. For the basis element y_L of $Y(A)$ in (3.3), y_LR is an analogous element of $Y(A_H)$ for the F-class $L \cap H = L \cap C$; if y_L is nilpotent so is y_LR. Since L is p-regular if and only if $L \cap H$ is, Lemma 1 implies the result.

The next theorem generalizes [3, Theorem 1], which in turn strengthens [4, Corollary 1] and [12, Corollary 2 to Theorem 9].

THEOREM 6. *For any p-subgroup D of G, the number of block idempotents of $Z(A)$ with defect group D is not greater than the number of p-regular A-nonnilpotent F-classes of G with defect group D.*

Proof. By Brauer's first main theorem on blocks, suitably generalized [5, (10B)], [17, Theorem 5.3], [14, Theorem 1] and by Lemma 2, we reduce at once to the case $G = N_G(D)$. In this case, let V be the F-subspace of $Z(A)$ with a basis consisting of the elements z_K (see the paragraph after (3.3)) for the $\boldsymbol{K}_A$-regular conjugacy classes K of G with defect group D. By [17, Lemmas 4.1 and 4.4], [8, p. 166] and [14, p. 281], V is a (commutative) subalgebra of $Z(A)$ (possibly without an identity) and the idempotents e mentioned in the statement are precisely the block idempotents of V. By Theorem

4 they are the block idempotents of $U = V \cap Y(A)$, which is a subalgebra of $Z(A)$ with a basis consisting of the elements y_L for the $\boldsymbol{D}_A$-regular F-classes L of G with defect group D. The block idempotents of $U/J(U)$ are the elements $e + J(U)$. Since these are linear combinations of the elements $y_L + J(U)$ for the F-classes L mentioned in the statement, the theorem is proved.

COROLLARY 1 (cf. [2, Lemma 1]). *The number of block idempotents of $Z(A)$ is not greater than the number of p-regular A-nonnilpotent F-classes of G.*

Theorem 6 and its proof, together with the theory of commutative algebras [11], yield the following corollaries, which generalize results of Brauer [5, (13A)] and Bovdi [3, Theorem 3] concerning the case $D = \{1\}$.

COROLLARY 2. *For any p-subgroup D of G, the number of block idempotents of $Z(A)$ with defect group D is the E-dimension of $U^E/J(U^E)$, where U is defined for D in $N_G(D)$. This equals the F-dimension of U^i for sufficiently large i.*

COROLLARY 3. *The following conditions are equivalent, where $H = N_G(D)$:*

(4.1) *There exists a block idempotent of $Z(A)$ with defect group D.*

(4.2) *There exists an A_H-nonnilpotent F-class of H with defect group D.*

(4.3) *There exists a p-regular A_H-nonnilpotent F-class of H with defect group D.*

Now we obtain some sufficient conditions for equality in Theorem 6. First we consider groups such that $O_{p,p',p}(G) = G$.

THEOREM 7 (cf. [2, Theorems 1 and 2], [3, Theorem 2]). *Assume that G has normal subgroups P and M, $P \subseteqq M$, such that P and G/M are p-groups while M/P is a p'-group. Then the number of block idempotents of $Z(A)$ is equal to the number of p-regular A-nonnilpotent F-classes of G. These coincide with the $\boldsymbol{D}_A$-regular F-classes of G which are contained in $O_{p'}(G)$, and also with the p-regular $\boldsymbol{D}_A$-regular F-classes of G with a defect group which contains P.*

Proof. By Burnside's theorem $Z(P)$ has a normal complement Q in $C = C_M(P)$. Then $C = Z(P) \times Q$, and easily $Q = O_{p'}(M) = O_{p'}(G)$.

Let L be any p-regular $\boldsymbol{D}_A$-regular F-class of G; then $L \subseteqq M$.

We claim that the following conditions on L are equivalent: (a) $L \subseteq Q$; (b) $L \subseteq C$; (c) L has a defect group which contains P; (d) the F-classes of M contained in L have defect group P; (e) L is A-nonnilpotent; (f) the conjugacy classes of M contained in L are A_M-nonnilpotent. It is straightforward that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d) $\Leftrightarrow$ (c). Since A_Q is semisimple [8, p. 156], (a) $\Rightarrow$ (e). Suppose now that (e) holds; let K_1 be a fixed conjugacy class of M contained in L. Then L is a disjoint union of classes of form $K = \{g\boldsymbol{d}_G(\sigma, x)\colon\ g \in K_1\}$ for suitable choices of $(\sigma, x) \in \mathscr{G} \times G$. For the element $y_L = \sum_{g \in L} q(g)a_g$ of (3.3), let $z_K = \sum_{g \in K} q(g)a_g$. Then $y_L = \sum z_K$, and z_K is a choice for the basis element of $Z(A_Q)$ corresponding to K. Since $y_L \boldsymbol{D}_A(\sigma, x) = y_L$, $z_{K_1}\boldsymbol{d}_G(\sigma, x) = z_K$. By (2.8) the elements z_K are either all nilpotent or all nonnilpotent; since their sum is nonnilpotent, so are they; hence (e) $\Rightarrow$ (f). Finally (f) $\Rightarrow$ (b) by the twisted generalization [8, p. 166] of [17, Lemma 4.2].

Let e be any block idempotent of $Z(A)$. Since the expression for e involves only p-regular elements, $e \in Z(A_M)$. By [15, Lemma 3], $e \in Z(A_C)$; then $e \in Z(A_Q)$ since (b) $\Rightarrow$ (a). (Alternatively: by the twisted generalization of [17, Proposition 4.4] which is implicit in [8, § 3], every block idempotent of $Z(A_M)$ has defect group P. The proof of Theorem 6 shows that e is in the algebra V defined for P in M; then $e \in Z(A_Q)$ since (d) $\Rightarrow$ (a).) Therefore the block idempotents of $Z(A)$ are identical with those of $Z(A) \cap Z(A_Q)$, and with those of $Y(A)^E \cap Y(A_Q)^E$. $Z(A_Q^E)$, being semisimple, is a direct sum of copies of E; then so is $Y(A)^E \cap Y(A_Q)^E$, and the number of block idempotents of $Z(A)$ equals the dimension of that algebra, namely the number of $\boldsymbol{D}_A$-regular F-classes of G which are contained in Q. Since (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e), the theorem is proved.

Together with [16, Theorem 6], Theorem 7 implies:

COROLLARY 4. *If G has a normal p-complement, each block of A contains exactly one irreducible representation of A over F.*

Combining Theorems 6 and 7 we obtain:

THEOREM 8 (cf. [3, Corollary 3]). *If G satisfies the hypothesis of Theorem 7, then for every p-subgroup D of G we have equality in Theorem 6.*

We conclude by treating the case of highest defect [5, (6D)], [17, Theorem 6.1], [8, p. 166], [3, Theorem 4], [10]. Our argument, based on [3], differs from that of [17] and [8] in using subalgebras of $Z(A_H)$ instead of a quotient algebra, and thus avoids counting p-singular classes.

THEOREM 9. *If P is a p-Sylow subgroup of G, the number of block idempotents of $Z(A)$ with defect group P is equal to the number of p-regular $\boldsymbol{D}_A$-regular F-classes of G with defect group P. All such F-classes are A-nonnilpotent.*

Proof. By the first main theorem on blocks, the number of block idempotents in question is equal to the number of block idempotents of $Z(A_H)$ with defect group P, where $H = N_G(P)$. These are all the block idempotents of $Z(A_H)$, as in the proof of Theorem 7; by that theorem, for H, the number of such block idempotents equals the number of p-regular $\boldsymbol{D}_{A_H}$-regular F-classes of H with defect group P. By the bijection of Lemma 1, this equals the number of the F-classes of G mentioned in the first sentence. The F-classes of H in question here are all A_H-nonnilpotent since (c) $\Rightarrow$ (e) in the proof of Theorem 7; then Lemma 2 implies the second sentence.

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Received October 10, 1970. This research was supported in part by the National Science Foundation under Grants GP-11424 and GP-19661.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 38, No. 1 March, 1971