Let $f$ be an entire function of a single complex variable. The exponential type of $f$ is given by

$$\tau(f) = \limsup_{n \to \infty} |f^{(n)}(0)|^{1/n}.$$ 

The Whittaker constant $W$ is defined to be the supremum of numbers $c$ having the following property: if $\tau(f) < c$ and each of $f, f', f'', \ldots$ has a zero in the disc $|z| \leq 1$, then $f \equiv 0$. The Whittaker constant is known to lie between .7259 and .7378.

The present paper provides a definition and characterization of the Whittaker constant $\mathcal{W}_n$ for $n$ complex variables. The principle result of this characterization, which involves polynomial expansions of entire functions, is

$$W > \mathcal{W}_2 \geq \mathcal{W}_3 \geq \cdots.$$ 

To simplify notation, the presentation here is given for functions of two variables.

An exact determination of $W$ was obtained by M. A. Evgrafov in 1954 [3]. The determination involves the Gončarov polynomials, defined recursively by

$$G_0(z) = 1,$$

(1.1) $$G_n(z; z_0, z_1, \cdots, z_{n-1}) = \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_0^{n-k}}{(n-k)!} G_k(z; z_0, z_1, \cdots, z_{k-1}).$$

Let

$$H_n = \max |G_n(0; z_0, \cdots, z_{n-1})|,$$

where the maximum is taken over all sequences $(z_k)_{k=0}^{n-1}$ whose terms lie on $|z| = 1$. Evgrafov proved that

$$W = \left\{ \limsup_{n \to \infty} H_n^{1/n} \right\}^{-1}.$$

An improvement of this result and further characterizations of $W$ were furnished by J. D. Buckholtz [1]. Using properties of the Gončarov polynomials, Buckholtz proved that

(1.2) $$(.4)^{1/n} H_n^{-1/n} < W \leq H_n^{-1/n},$$

for $n = 1, 2, 3, \cdots$. A consequence of these bounds is
For an entire function \( f \) (of two complex variables) the exponential type \( \tau(f) \) is given by

\[
\tau(f) = \limsup_{m+n \to \infty} |f^{(m,n)}(0,0)|^{1/(m+n)}.
\]

We define the Whittaker constant \( \mathcal{W} \) to be the supremum of positive numbers \( c \) having the following property: if \( \tau(f) < c \) and each of \( f^{(m,n)} \) (\( 0 \leq m < \infty \), \( 0 \leq n < \infty \)) has a zero in the poly disc \( \{ (z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1 \} \), then \( f \equiv 0 \). The bound \( \mathcal{W} \geq (\log 2)/2 \) was obtained by M. M. Dzrbasjan in 1957 [2].

The estimate furnished by Dzrbasjan depends on a system of polynomials defined as follows. Let \( \alpha = (\alpha_{pq}) \) and \( \beta = (\beta_{pq}) \) be infinite matrices of complex numbers. The polynomials \( A_{m,n}(z_1, z_2; \alpha, \beta) \) are defined by the recursion formula

\[
A_{0,0}(z_1, z_2) = 1,
\]

(1.4) \[ A_{r,s}(z_1, z_2; \alpha, \beta) = \frac{z_1^r z_2^s}{r! s!} - \sum_{p+q \leq r+s} \sum_{p \geq q} \frac{A_{p,q}(z_1, z_2; \alpha, \beta) \alpha_{pq} \beta_{pq}^{r-q}}{(r-p)! (s-q)!} \]

for \( r, s = 0, 1, 2, \ldots \). Note that \( A_{r,s} \) depends only on those parameters \( \alpha_{pq} \) and \( \beta_{pq} \) for which \( p + q < r + s \). Let

\[
H_{r,s} = \max |A_{r,s}(0, 0; \alpha, \beta)|,
\]

where the maximum is taken over all matrices \( \alpha \) and \( \beta \) whose entries lie on \( |z| = 1 \). We show that bound \( H_{r,s} \leq (2/\log 2)^{r+s} \) holds for all \( r \) and \( s \). The justifies the definition

\[
H = \sup_{1 \leq r, s < \infty} H_{r,s}^{1/(r+s)}.
\]

We prove the following expansion theorem.

**Theorem 1.** Suppose \( f \) is entire and \( \tau(f) < 1/H \). If \( \alpha \) and \( \beta \) are infinite complex matrices whose entries lie in \( |z| \leq 1 \), then

\[
f(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f^{(m,n)}(\alpha_{mn}, \beta_{mn}) A_{m,n}(z_1, z_2; \alpha, \beta)
\]

for all \( (z_1, z_2) \).

The following result shows that the expansion constant \( 1/H \) is as large as possible.

**Theorem 2.** There exists an entire function \( F \), with \( \tau(F) = \)
Theorem 1 and Theorem 2 will be proved in § 3. We note, however, that the following result is an easy consequence of Theorems 1 and 2.

**Corollary 1.** $\Psi^- = 1/H$.

Therefore, each of the numbers $H_{m,n}^{-1}$ is an upper bound for $\Psi$. In particular, $\Psi^- \leq 1/\sqrt{H_{1,1}} = 1/\sqrt{3}$. In comparing this with the bound $W > .7259$, one sees that $\Psi^- < W$.

2. The Polynomials $A_{m,n}$. Let $f$ be an entire function and let $\alpha$ and $\beta$ be infinite complex matrices. Writing (1.4) in the form

$$
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{A_{p,q}(z_1, z_2; \alpha, \beta) \alpha_{p}^{r-p} \beta_{p}^{s-q}}{(r-p)! (s-q)!}
$$

we obtain the formal expansion

$$
f(z_1, z_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(r,s)}(0,0) \frac{x_1^r x_2^s}{r! s!}
$$

which holds whenever the interchange in the order of summation can be justified. In particular, (2.1) holds if $f$ is a polynomial and yields considerable information when $f$ is taken to be one of the polynomials $A_{m,n}$.

**Lemma 1.** If $\lambda$ is a complex number, then

$$A_{m,n}(\lambda z_1, \lambda z_2; \lambda \alpha, \lambda \beta) = \lambda^{m+n} A_{m,n}(z_1, z_2; \alpha, \beta),$$

where $\lambda \alpha$ denotes matrix scalar multiplication. Furthermore,

$$A_{m,n}(\alpha_{\infty}, \beta_{\infty}; \alpha, \beta) = 0 \quad (m + n > 0).$$

**Proof.** We will prove (2.2) using mathematical induction. The proof of (2.3) is similar. If $m + n = 0$, the result is clear. Suppose
$N$ is a positive integer and (2.2) holds for the polynomials $A_{p,q}$ with $p + q < N$. If $r$ and $s$ are nonnegative integers such that $r + s = N$, then

$$A_{r,s}(\lambda z_1, \lambda z_2; \lambda \alpha, \lambda \beta)$$

$$= \frac{\lambda^{r+s} z_1^{r} z_2^{s}}{r! s!} - \sum_{p=0}^{r} \sum_{q=0}^{s} \frac{A_{p,q}(\lambda z_1, \lambda z_2; \lambda \alpha, \lambda \beta)(\lambda \alpha_{pq})^{r-p}(\lambda \beta_{pq})^{s-q}}{(r-p)! (s-q)!}$$

$$= \lambda^{r+s} A_{r,s}(z_1, z_2; \alpha, \beta)$$

and this completes the proof.

Let $\alpha = (\alpha_{pq})_{p,q=0}^{\infty}$ be an infinite complex matrix. If $j$ and $k$ are nonnegative integers, we denote by $R_{jk}$ the operator which transforms $\alpha$ into

$$R_{jk}(\alpha) = (\alpha_{p+j, q+k})_{p,q=0}^{\infty}.$$  

**Lemma 2.** If $m + n > 0$, $j \leq m$ and $k \leq n$, then

(2.4) \hspace{1cm} A_{m,n}^{j,k}(z_1, z_2; \alpha, \beta) = A_{m-j, n-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta)).$

**Proof.** By direct computation, $A_{0,0}(z_1, z_2; \alpha, \beta) = z_1 - \alpha_{00}$ and

$$A_{0,j}(z_1, z_2; \alpha, \beta) = z_2 - \beta_{00},$$

so the result is clear if $m + n = 1$. Proceeding inductively, let $N$ be a positive integer and suppose the proposition is true for the polynomials $A_{p,q}$ with $p + q < N$. If $r$ and $s$ are nonnegative integers such that $r + s = N$, then for $j \leq r$ and $k \leq s$ we have

$$A_{r,s}^{j,k}(z_1, z_2; \alpha, \beta)$$

$$= \frac{z_1^{r-j} z_2^{s-k}}{(r-j)! (s-k)!} - \sum_{p=0}^{r-j} \sum_{q=0}^{s-k} \frac{A_{p,q}^{j,k}(z_1, z_2; \alpha, \beta)(\alpha_{pq})^{r-j-p}(\beta_{pq})^{s-k-q}}{(r-j-p)! (s-k-q)!}$$

$$= \frac{z_1^{r-j} z_2^{s-k}}{(r-j)! (s-k)!} - \sum_{p=0}^{r-j} \sum_{q=0}^{s-k} \frac{A_{p,q}^{j,k}(z_1, z_2; \alpha, \beta)\alpha_{p+q}^{r-j-p}(\beta_{p+q})^{s-k-q}}{(r-j-p)! (s-k-q)!}$$

$$= A_{r-j, s-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta)),$$

and this completes the proof.

Lemma 2 and the expansion (2.1) provide a useful expression for the polynomials $A_{m,n}$. Replacing $\alpha$ and $\beta$ by $\gamma$ and $\delta$, respectively,
and applying (2.1) to the polynomial \( A_{r,s}(z_1, z_2; \alpha, \beta) \), we have

\[
A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^{r} \sum_{q=0}^{s} A_{p,q}^{(r,s)}(\gamma_{pq}, \delta_{pq}; \alpha, \beta) A_{r-p, s-q}(\gamma_{pq}, \delta_{pq}; R_{pq}(\alpha), R_{pq}(\beta)).
\]

(2.5)

If each of \( \gamma \) and \( \delta \) is the zero matrix, it is easy to see that

\[
A_{p,q}(z_1, z_2; \gamma, \delta) = \frac{z_1^p z_2^q}{p! q!}.
\]

In this case (2.5) yields

\[
A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^{r} \sum_{q=0}^{s} A_{r-p, s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) \frac{z_1^p z_2^q}{p! q!}.
\]

(2.6)

Let \( m \) and \( n \) be integers such that \( 0 \leq m \leq r \), \( 0 \leq n \leq s \), and \( m + n > 0 \). In (2.5) choose

\[
\gamma_{pq} = \begin{cases} 0, & \text{if } p \geq m \text{ and } q \geq n \\ \alpha_{pq}, & \text{otherwise} \end{cases}
\]

and

\[
\delta_{pq} = \begin{cases} 0, & \text{if } p \geq m \text{ and } q \geq n \\ \beta_{pq}, & \text{otherwise} \end{cases}.
\]

In view of (2.3) we have

\[
A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=m}^{r} \sum_{q=n}^{s} A_{p,q}(z_1, z_2; \gamma, \delta) A_{r-p, s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)).
\]

(2.7)

More generally, we define the operator \( P_{jk} \) as follows. If \( j + k > 0 \), then \( P_{jk}(\alpha) \) is the matrix \( (a_{pq}) \), where

\[
a_{pq} = \begin{cases} 0, & \text{if } p \geq j \text{ and } q \geq k \\ \alpha_{pq}, & \text{otherwise} \end{cases}.
\]

Then (2.7) becomes

\[
A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=m}^{r} \sum_{q=n}^{s} A_{p,q}(z_1, z_2; P_{mn}(\alpha), P_{mn}(\beta)) A_{r-p, s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)).
\]

(2.8)

Equation (2.8) may be regarded as a separation of variables formula, in the following sense. If \( p \geq m \) and \( q \geq n \), then \( R_{pq}(\alpha) \) depends on the parameters \( \alpha_{jk} \), where \( j \geq m \) and \( k \geq m \), and \( P_{mn}(\alpha) \) depends
on the parameters $\alpha_{jk}$, where $j < m$ or $k < n$. The usefulness of (2.8) is seen in the next lemma.

**Lemma 3.** If $0 \leq m \leq r$ and $0 \leq n \leq s$, then

$$H_{r,s} \geq H_{m,n} H_{r-m,s-n} \quad \ldots \quad (2.9)$$

*Proof.* If $m + n = 0$, the result is trivial. Suppose $m + n > 0$ and choose matrices $\alpha$ and $\beta$, whose entries lie on $|z| = 1$, such that

$$H_{m,n} = |A_{m,n}(0, 0; P_{mn}(\alpha), P_{mn}(\beta))|$$

and

$$H_{r-m,s-n} = |A_{r-m,s-n}(0, 0; R_{mn}(\alpha), R_{mn}(\beta))| .$$

For each complex number $\lambda$, define the matrices $\gamma = \gamma(\lambda)$ and $\delta = \delta(\lambda)$ by

$$\gamma_{pq} = \begin{cases} \alpha_{pq}, & \text{if } p \geq m \text{ and } q \geq n \\ \lambda \alpha_{pq}, & \text{otherwise} \end{cases}$$

and

$$\delta_{pq} = \begin{cases} \beta_{pq}, & \text{if } p \geq m \text{ and } q \geq n \\ \lambda \beta_{pq}, & \text{otherwise} \end{cases} .$$

By (2.8) and (2.2),

$$A_{r,s}(0, 0; \gamma, \delta) = \sum_{p=m}^{r} \sum_{q=n}^{r} A_{p,q}(0, 0; P_{mn}(\gamma), P_{mn}(\delta)) A_{r-p,s-q}(0, 0; R_{pq}(\gamma), R_{pq}(\delta))$$

$$= \sum_{p=m}^{r} \sum_{q=n}^{s} \lambda^{p+q} A_{p,q}(0, 0; P_{mn}(\alpha), P_{mn}(\beta)) A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta))$$

$$= \lambda^{r+s} Q(\lambda) ,$$

where $Q(\lambda)$ is a polynomial in $\lambda$. Since

$$H_{r,s} \geq \max_{|z|=1} |A_{r,s}(0, 0; \gamma, \delta)| = \max_{|\lambda|=1} |Q(\lambda)| \geq |Q(0)|$$

and

$$|Q(0)| = |A_{m,n}(0, 0; P_{mn}(\alpha), P_{mn}(\beta))| \geq |A_{r-m,s-n}(0, 0; R_{mn}(\alpha), R_{mn}(\beta))|$$

$$= H_{m,n} H_{r-m,s-n} ,$$

we have

$$H_{r,s} \geq H_{m,n} H_{r-m,s-n} .$$
Lemma 4. There is an infinite subsequence $S = \{(m_j, n_j): j = 1, 2, 3, \ldots\}$ such that

(i) \[ H = \lim_{j \to \infty} H_{m_j, n_j}^{j(m_j + n_j)} \]

and

(ii) \[ H_{m_j, n_j}^{j(m_j + n_j)} \geq H_{p, q}^{j(p + q)} \]

for all $p$ and $q$ such that $p + q \leq m_j + n_j$.

Proof. If there is a pair $(r, s)$ such that $H_{r, s}^{j(r + s)} = H$, then (2.9) implies

\[ H \geq H_{r, s}^{j(r + s)} \geq (H_{r, s}^{j(r + s)})^{1/j(r + s)} = H_{r, s}^{j(r + s)} = H \]

for $j = 1, 2, 3, \ldots$. In this case we take $S = \{(jr, js): j = 1, 2, 3, \ldots\}$.

Suppose, on the other hand, that $H > H_{r, s}^{j(r + s)}$ for all $r$ and $s$. For each positive integer $k$, let

\[ T_k = \max_{p + q = k} H_{p, q}^{j(p + q)} \]

Then $T_k < H(1 \leq k < \infty)$ and $\sup_{1 \leq k < \infty} T_k = H$. We can therefore find a subsequence $\{T_{k_j}\}_{j=1}^{\infty}$ with the properties that

\[ \lim_{j \to \infty} T_{k_j} = H \]

and

\[ T_{k_j} > T_n \]

for $n < k_j$. For each $j$, choose integers $m_j$ and $n_j$ such that $m_j + n_j = k_j$ and $T_{k_j} = H_{m_j, n_j}^{j(m_j + n_j)}$, and let $S = \{(m_j, n_j): j = 1, 2, 3, \ldots\}$. This completes the proof of the lemma.

Corollary 2. $H = \lim_{m + n \to \infty} \sup_{m, n} H_{m, n}^{j(m + n)}$.

Lemma 5. For each pair of nonnegative integers $(m, n)$ we have

(2.10) \[ H_{m, n} \leq (2/\log 2)^{m+n} \]

Proof. The result is trivial if $m + n = 0$. Let $N$ be a positive integer and suppose (2.10) holds whenever $m + n < N$. Let $r$ and $s$ be nonnegative integers such that $r + s = N$. The defining relations (1.4) imply
\[ H_{r,s} \leq \sum_{p=0}^{r} \sum_{q=0}^{s} \frac{H_{p,q}}{(r-p)!(s-q)!} = \sum_{j=0}^{r} \sum_{k=0}^{s} \frac{H_{r-j,s-k}}{j!k!} \]
\[ \leq \sum_{j=0}^{r} \sum_{k=0}^{s} \frac{(2/\log 2)^{j+k}}{j!k!} = (2/\log 2)^{r+s}(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2/\log 2)^{j+k}}{j!k!} - 1) \]
\[ = (2/\log 2)^{r+s}(e^{(2/\log 2)^2} - 1) = (2/\log 2)^{r+s}. \]

**Corollary 3.** \( H \leq (2/\log 2). \)

Note that this result, together with Corollary 1, implies Džrbašjan's estimate \( \gamma^r \geq (\log 2)/2. \)

3. **Main Results.** Let
\[ M(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{H_{p,q}} \frac{z_1^p z_2^q}{p!q!}. \]

Note that \( M(z_1, z_2) \) is an entire function of exponential type 1 or less. Suppose \( \alpha \) and \( \beta \) have entries lying in \( |z| \leq 1. \) By (2.6),
\[ A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^{r} \sum_{q=0}^{s} A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) \frac{z_1^p z_2^q}{p!q!}. \]

Since
\[ |A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta))| \leq H_{r-p,s-q} \leq H_{r,s}/H_{p,q}, \]
it follows that the coefficients of \( A_{r,s} \) are bounded by the respective coefficients of \( H_{r,s}M(z_1, z_2); \) i.e., \( A_{r,s} \) is majorized by \( H_{r,s}M(z_1, z_2). \) In particular,
\[ |A_{r,s}(z_1, z_2; \alpha, \beta)| \leq H_{r,s}M(|z_1|, |z_2|). \]

We are now ready to prove Theorem 1.

Suppose \( f \) is an entire function, with \( \tau(f) < 1/H, \) and suppose \( \alpha \) and \( \beta \) are matrices whose entries lie in \( |z| \leq 1. \) In order to justify the expansion (2.1) we show that the series
\[ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} |f^{(r,s)}(0, 0)| \sum_{p=0}^{r} \sum_{q=0}^{s} \frac{|A_{p,q}(z_1, z_2; \alpha, \beta)|}{(r-p)!(s-q)!} \]
is convergent. Equation (3.1) implies
\[ |A_{p,q}(z_1, z_2; \alpha, \beta)| \leq H_{p,q} M(|z_1|, |z_2|) \leq H_{r,s} M(|z_1|, |z_2|) / H_{r-p, s-q}; \]

therefore

\[ \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{|A_{p,q}(z_1, z_2; \alpha, \beta)|}{(r - p)! (s - q)!} \leq H_{r,s} M(|z_1|, |z_2|) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{H_{r-p, s-q}(r - p)! (s - q)!} < H_{r,s} M(|z_1|, |z_2|) M(1, 1). \]

The series (3.2) is therefore convergent provided that

\[ \sum_{r+s \geq N} |f^{(r,s)}(0,0)| H_{r,s}, \]

converges. Choose \( \varepsilon > 0 \) such that \( \tau(f) + \varepsilon < 1/H \) and let \( N \) be a positive integer such that \( r + s \geq N \) implies

\[ |f^{(r,s)}(0,0)|^{1/(r+s)} < \tau(f) + \varepsilon. \]

Then

\[ \sum_{r+s \geq N} |f^{(r,s)}(0,0)| H_{r,s} \leq \sum_{r+s \geq N} [H(\tau(f) + \varepsilon)]^{r+s}. \]

Let \( \rho = H(\tau(f) + \varepsilon) \) and \( K = \sum_{r+s < N} |f^{(r,s)}(0,0)| H_{r,s}. \) Then (3.3) is less than

\[ K + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \rho^{r+s} = K + \frac{1}{(1 - \rho)^2} \]

and the convergence of (3.2) follows.

**Proof of Theorem 2.** Let \( S = \{(m_j, n_j): j = 1, 2, 3, \ldots \} \) be an infinite sequence such that

\[ H = \lim_{j \to \infty} H^{1/(m_j + n_j)} \]

and

\[ H^{1/(m_j + n_j)} \geq H_{p,q}^{1/(p+q)} \]

for all \( p \) and \( q \) such that \( p + q \leq m_j + n_j. \) For each \( (r, s) \in S, \) let \( \alpha = \alpha(r, s) \) and \( \beta = \beta(r, s) \) be matrices with entries on \( |z| = 1 \) such that

\[ |A_{r,s}(0,0; \alpha, \beta)| = H_{r,s}. \]

Let
\[ P_{r,s}(z_1, z_2) = \frac{A_{r,s}(z_1, z_2; \alpha, \beta)}{A_{r,s}(0, 0; \alpha, \beta)} \]

and

\[ Q_{r,s}(z_1, z_2) = P_{r,s} \left( \frac{z_1 H^{(r+s)}_{r,s}}{H}, \frac{z_2 H^{(r+s)}_{r,s}}{H} \right) \].

Then \( Q_{r,s}(0, 0) = P_{r,s}(0, 0) = 1 \), and

\[ Q^{(j,k)}(r,s) \left( \frac{H \alpha_{j,k}}{H^{(r+s)}_{r,s}}, \frac{H \beta_{j,k}}{H^{(r+s)}_{r,s}} \right) = 0 \quad (j < r, k < s) \]

Moreover, (2.6) implies

\[ Q_{r,s}(z_1, z_2) = \sum_{p=0}^{r} \sum_{q=0}^{s} A_{r-p,s-q}(0, 0; R_{pq}^1(\alpha), R_{pq}^1(\beta)) \frac{H^{(p+q)/(r+s)}}{p! q!} \]

and

\[ \left| \frac{A_{r-p,s-q}(0, 0; R_{pq}^1(\alpha), R_{pq}^1(\beta)) H^{(p+q)/(r+s)}}{A_{r,s}(0, 0; \alpha, \beta) H^{p+q}} \right| \leq \frac{H^{(r-p+s-q)/(r+s)} H^{(p+q)/(r+s)}}{H^{r,s} H^{p+q}} = \frac{1}{H^{p+q}} \]

since \((r, s) \in S\). Therefore \( Q_{r,s} \) is majorized by

\[ \varphi(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{H^{p+q}} \frac{z_1^p z_2^q}{p! q!} \]

\( \varphi(z_1, z_2) \) is an entire function of exponential type \( 1/H \). The sequence \( \{Q_{m,j,n}\} \) is therefore uniformly bounded on compact sets. Extract a uniformly convergent subsequence from \( \{Q_{m,j,n}\} \) and let \( F \) denote the limit function. Then \( F \) is entire, \( F(0, 0) = 1 \), and \( \tau(F) \leq 1/H \).

Since \( F^{(j,k)} \) is the uniform limit of a subsequence of \( \{Q_{m,j,n}\} \), then (3.4) implies that \( F^{(j,k)} \) has a zero in \( \{|z_1| = 1, |z_2| = 1\} \). The expansion (1.5) implies that \( F \) has exponential type exactly \( 1/H \), and this completes the proof.

### 4. The Whittaker Constants \( \mathcal{W} \) and \( \mathcal{W}_\mathcal{W} \)

We have already seen that \( \mathcal{W} < W \). The following result provides a precise relationship between \( \mathcal{W} \) and \( W \), and a determination of \( W \) different from [3] and [1].

**Theorem 3.** \[ \limsup_{m \to \infty} H^{(m+n)}_{m+n} = 1/\mathcal{W} \quad , \]

\[ \liminf_{m \to \infty} H^{(m+n)}_{m+n} = 1/W \]
Proof. The first equation is a consequence of Corollary 1 and Corollary 2. To prove the second, we require the use of the Gončarov polynomials $G_n(z; z_0, \ldots, z_{n-1})$ and the sequence 

$$H_n = \max |G_n(0; z, \ldots, z_{n-1})|.$$ 

If $m$ is a positive integer, the defining relation (1.4) implies

$$A_{m,0}(0, 0; \alpha, \beta) = -\sum_{p=0}^{m-1} \frac{A_{p,0}(0, 0; \alpha, \beta)\alpha^{m-p}}{(m-p)!}.$$ 

In comparing (4.1) with (1.1), one sees that 

$$A_{m,0}(0, 0; \alpha, \beta) = G_m(0; \alpha_0, \alpha_1, \ldots, \alpha_m).$$ 

It follows that $H_{m,0} = H_m$ and, similarly, $H_{0,m} = H_m$. By Lemma 3 and (1.2), we have

$$H_{m,n} \geq (H_{m,0}H_{0,n})^{1/(m+n)} = (H_mH_n)^{1/(m+n)} > \left(\frac{.16}{W^{m+n}}\right)^{1/(m+n)} = \frac{(.16)^{1/(m+n)}}{W}.$$ 

Therefore

$$\liminf_{m,n \to \infty} H_{m,n}^{1/(m+n)} \geq 1/W.$$ 

In the other direction,

$$\liminf_{m,n \to \infty} H_{m,n}^{1/(m+n)} \leq \liminf_{m \to \infty} H_{m,0}^{1/(m+0)} = \liminf_{m \to \infty} H_m^{1/m} = 1/W,$$

and this completes the proof.

Using (2.10) and the estimate $W < .7378$, one easily obtains an interesting bound on $\mathcal{W}$. For all $r$ and $s$, we have

$$H_{r,s} \leq (2/\log 2)^{r+s} \left(\frac{2}{\log 2} \cdot .7378\right)^{r+s} < \left(\frac{2.13}{W}\right)^{r+s},$$

and therefore

$$W > \mathcal{W} \geq \frac{W}{2.13}.$$ 

Some remarks should be made relative to stating the above results in terms of $k$ complex variables, $k > 2$. For $j = 1, 2, \ldots, k$, let $\alpha^{(j)} = (\alpha^{(j)}_{n_1, n_2, \ldots, n_k})$ denote a $k$-parameter sequence of complex numbers. The recursion relation corresponding to (1.4) is

$$A_{0,0,\ldots,0}(z_1, z_2, \ldots, z_k) = 1$$

and
\[ \begin{align*}
A_{n_1,\ldots,n_k}(z_1, z_2, \ldots, z_k) &= z_1^{n_1} \cdots z_k^{n_k} - \sum_{p_1=0}^{n_1} \cdots \sum_{p_k=0}^{n_k} \frac{A_{p_1,\ldots,p_k}(z_1, \ldots, z_k) [\alpha_{p_1}(\ldots)\alpha_{p_k}(\ldots)]^{n_1-p_1} \cdots [\alpha_{p_1}(\ldots)\alpha_{p_k}(\ldots)]^{n_k-p_k}}{(n_i - p_i)! \cdots (n_k - p_k)!} \\
\text{where } p_1 + \cdots + p_k &< n_i + \cdots + n_k.
\end{align*} \]

The numbers \( H_{n_1,\ldots,n_k} \) are also defined in the obvious way and we have

\[ H_{n_1,\ldots,n_k} \geq H_{m_1,\ldots,m_k} H_{n_1-m_1,\ldots,n_k-m_k}, \]
\[ H_{n_1,\ldots,n_k,0,\ldots,0} = H_{n_1,\ldots,n_k}. \]

The definition of \( W_k \), the Whittaker constant in \( k \) complex variables, is analogous to the definition of \( W \) in §1. Apart from notational difficulties, it is a direct extension of the above results to see that

\[ \lim \sup H_{n_1,\ldots,n_k}^{(n_1+\cdots+n_k)} = \frac{1}{W_k}, \]

and

\[ \lim \inf H_{n_1,\ldots,n_k}^{(n_1+\cdots+n_k)} = \frac{1}{W_k}. \]

If \( 1 \leq l \leq k \), we also have

\[ \lim \sup H_{n_1,\ldots,n_l,0,\ldots,0}^{(n_1+\cdots+n_l)} = \frac{1}{W_l}, \]

and

\[ \lim \inf H_{n_1,\ldots,n_l,0,\ldots,0}^{(n_1+\cdots+n_l)} = 1/W, \]

and it follows that \( W = W_2 \geq W_3 \geq W_4 \geq \cdots. \)

\section*{References}


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