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**WHITTAKER CONSTANTS FOR ENTIRE FUNCTIONS OF
SEVERAL COMPLEX VARIABLES**

KEN SHAW

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JOHN K. SHAW

Let f be an entire function of a single complex variable. The exponential type of f is given by

$$\tau(f) = \limsup_{n \rightarrow \infty} |f^{(n)}(0)|^{1/n}.$$

The Whittaker constant W is defined to be the supremum of numbers c having the following property: if $\tau(f) < c$ and each of f, f', f'', \dots has a zero in the disc $|z| \leq 1$, then $f \equiv 0$. The Whittaker constant is known to lie between .7259 and .7378.

The present paper provides a definition and characterization of the Whittaker constant \mathcal{W}_n for n complex variables. The principle result of this characterization, which involves polynomial expansions of entire functions, is

$$W > \mathcal{W}_2 \geq \mathcal{W}_3 \geq \dots$$

To simplify notation, the presentation here is given for functions of two variables.

An exact determination of W was obtained by M. A. Evgrafov in 1954 [3]. The determination involves the Gončarov polynomials, defined recursively by

$$G_0(z) = 1,$$

$$(1.1) \quad G_n(z; z_0, z_1, \dots, z_{n-1}) = \frac{z^n}{n!} - \sum_{k=0}^{n-1} \frac{z_k^{n-k}}{(n-k)!} G_k(z; z_0, z_1, \dots, z_{k-1}).$$

Let

$$H_n = \max |G_n(0; z_0, \dots, z_{n-1})|,$$

where the maximum is taken over all sequences $\{z_k\}_{k=0}^{n-1}$ whose terms lie on $|z| = 1$. Evgrafov proved that

$$W = \left\{ \limsup_{n \rightarrow \infty} H_n^{1/n} \right\}^{-1}.$$

An improvement of this result and further characterizations of W were furnished by J. D. Buckholtz [1]. Using properties of the Gončarov polynomials, Buckholtz proved that

$$(1.2) \quad (.4)^{1/n} H_n^{-1/n} < W \leq H_n^{-1/n},$$

for $n = 1, 2, 3, \dots$. A consequence of these bounds is

$$(1.3) \quad W = \left\{ \lim_{n \rightarrow \infty} H_n^{1/n} \right\}^{-1} = \left\{ \sup_{1 \leq n < \infty} H_n^{1/n} \right\}^{-1}.$$

For an entire function f (of two complex variables) the exponential type $\tau(f)$ is given by

$$\tau(f) = \limsup_{m+n \rightarrow \infty} |f^{(m,n)}(0, 0)|^{1/(m+n)}.$$

We define the Whittaker constant \mathscr{W} to be the supremum of positive numbers c having the following property: if $\tau(f) < c$ and each of $f^{(m,n)}$ ($0 \leq m < \infty, 0 \leq n < \infty$) has a zero in the poly disc $\{(z_1, z_2): |z_1| \leq 1, |z_2| \leq 1\}$, then $f \equiv 0$. The bound $\mathscr{W} \geq (\log 2)/2$ was obtained by M. M. Džrbašjan in 1957 [2].

The estimate furnished by Džrbašjan depends on a system of polynomials defined as follows. Let $\alpha = (\alpha_{pq})$ and $\beta = (\beta_{pq})$ be infinite matrices of complex numbers. The polynomials $A_{m,n}(z_1, z_2; \alpha, \beta)$ are defined by the recursion formula

$$(1.4) \quad A_{0,0}(z_1, z_2) = 1, \\ A_{r,s}(z_1, z_2; \alpha, \beta) = \frac{z_1^r z_2^s}{r!s!} - \sum_{\substack{p=0 \\ p+q < r+s}}^r \sum_{q=0}^s \frac{A_{p,q}(z_1, z_2; \alpha, \beta) \alpha_{pq}^{r-p} \beta_{pq}^{s-q}}{(r-p)!(s-q)!}$$

for $r, s = 0, 1, 2, \dots$. Note that $A_{r,s}$ depends only on those parameters α_{pq} and β_{pq} for which $p + q < r + s$. Let

$$H_{r,s} = \max |A_{r,s}(0, 0; \alpha, \beta)|,$$

where the maximum is taken over all matrices α and β whose entries lie on $|z| = 1$. We show that bound $H_{r,s} \leq (2/\log 2)^{r+s}$ holds for all r and s . This justifies the definition

$$H = \sup_{1 \leq r, s < \infty} H_{r,s}^{1/(r+s)}.$$

We prove the following expansion theorem.

THEOREM 1. *Suppose f is entire and $\tau(f) < 1/H$. If α and β are infinite complex matrices whose entries lie in $|z| \leq 1$, then*

$$(1.5) \quad f(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f^{(m,n)}(\alpha_{mn}, \beta_{mn}) A_{m,n}(z_1, z_2; \alpha, \beta)$$

for all (z_1, z_2) .

The following result shows that the expansion constant $1/H$ is as large as possible.

THEOREM 2. *There exists an entire function F , with $\tau(F) =$*

$1/H$, such that each of $F^{(m,n)}$ ($0 \leq m < \infty, 0 \leq n < \infty$) has a zero in the polydisc $\{|z_1| \leq 1, |z_2| \leq 1\}$.

Theorem 1 and Theorem 2 will be proved in §3. We note, however, that the following result is an easy consequence of Theorems 1 and 2.

COROLLARY 1. $\mathscr{W} = 1/H$.

Therefore, each of the numbers $H_{m,n}^{-1/(m+n)}$ is an upper bound for \mathscr{W} . In particular, $\mathscr{W} \leq 1/\sqrt{H_{1,1}} = 1/\sqrt{3}$. In comparing this with the bound $W > .7259$, one sees that $\mathscr{W} < W$.

2. The Polynomials $A_{m,n}$. Let f be an entire function and let α and β be infinite complex matrices. Writing (1.4) in the form

$$\frac{z_1^r z_2^s}{r! s!} = \sum_{p=0}^r \sum_{q=0}^s \frac{A_{p,q}(z_1, z_2; \alpha, \beta) \alpha_{pq}^{r-p} \beta_{pq}^{s-q}}{(r-p)! (s-q)!}$$

we obtain the formal expansion

$$\begin{aligned} f(z_1, z_2) &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(r,s)}(0, 0) \frac{z_1^r z_2^s}{r! s!} \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} f^{(r,s)}(0, 0) \left\{ \sum_{p=0}^r \sum_{q=0}^s \frac{A_{p,q}(z_1, z_2; \alpha, \beta) \alpha_{pq}^{r-p} \beta_{pq}^{s-q}}{(r-p)! (s-q)!} \right\} \\ (2.1) \quad &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{p,q}(z_1, z_2; \alpha, \beta) \left\{ \sum_{r=p}^{\infty} \sum_{s=q}^{\infty} f^{(r,s)}(0, 0) \frac{\alpha_{pq}^{r-p} \beta_{pq}^{s-q}}{(r-p)! (s-q)!} \right\} \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} f^{(p,q)}(\alpha_{pq}, \beta_{pq}) A_{p,q}(z_1, z_2; \alpha, \beta), \end{aligned}$$

which holds whenever the interchange in the order of summation can be justified. In particular, (2.1) holds if f is a polynomial and yields considerable information when f is taken to be one of the polynomials $A_{m,n}$.

LEMMA 1. If λ is a complex number, then

$$(2.2) \quad A_{m,n}(\lambda z_1, \lambda z_2; \lambda \alpha, \lambda \beta) = \lambda^{m+n} A_{m,n}(z_1, z_2; \alpha, \beta),$$

where $\lambda \alpha$ denotes matrix scalar multiplication. Furthermore,

$$(2.3) \quad A_{m,n}(\alpha_{00}, \beta_{00}; \alpha, \beta) = 0 \quad (m+n > 0).$$

Proof. We will prove (2.2) using mathematical induction. The proof of (2.3) is similar. If $m+n=0$, the result is clear. Suppose

N is a positive integer and (2.2) holds for the polynomials $A_{p,q}$ with $p+q < N$. If r and s are nonnegative integers such that $r+s = N$, then

$$\begin{aligned} & A_{r,s}(\lambda z_1, \lambda z_2; \lambda \alpha, \lambda \beta) \\ &= \lambda^{r+s} \frac{z_1^r z_2^s}{r! s!} - \sum_{\substack{p=0 \\ p+q < r+s}}^r \sum_{q=0}^s \frac{A_{p,q}(\lambda z_1, \lambda z_2; \lambda \alpha, \lambda \beta) (\lambda \alpha_{pq})^{r-p} (\lambda \beta_{pq})^{s-q}}{(r-p)! (s-q)!} \\ &= \lambda^{r+s} \frac{z_1^r z_2^s}{r! s!} - \lambda^{r+s} \sum_{\substack{p=0 \\ p+q < r+s}}^r \sum_{q=0}^s \frac{A_{p,q}(z_1, z_2; \alpha, \beta) \alpha_{pq}^{r-p} \beta_{pq}^{s-q}}{(r-p)! (s-q)!} \\ &= \lambda^{r+s} A_{r,s}(z_1, z_2; \alpha, \beta) \end{aligned}$$

and this completes the proof.

Let $\alpha = (\alpha_{pq})_{p,q=0}^{\infty}$ be an infinite complex matrix. If j and k are nonnegative integers, we denote by R_{jk} the operator which transforms α into

$$R_{jk}(\alpha) = (\alpha_{p+j, q+k})_{p,q=0}^{\infty}.$$

LEMMA 2. If $m+n > 0$, $j \leq m$ and $k \leq n$, then

$$(2.4) \quad A_{m,n}^{(j,k)}(z_1, z_2; \alpha, \beta) = A_{m-j, n-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta)).$$

Proof. By direct computation, $A_{1,0}(z_1, z_2; \alpha, \beta) = z_1 - \alpha_{00}$ and

$$A_{0,1}(z_1, z_2; \alpha, \beta) = z_2 - \beta_{00},$$

so the result is clear if $m+n = 1$. Proceeding inductively, let N be a positive integer and suppose the proposition is true for the polynomials $A_{p,q}$ with $p+q < N$. If r and s are nonnegative integers such that $r+s = N$, then for $j \leq r$ and $k \leq s$ we have

$$\begin{aligned} & A_{r,s}^{(j,k)}(z_1, z_2; \alpha, \beta) \\ &= \frac{z_1^{r-j} z_2^{s-k}}{(r-j)! (s-k)!} - \sum_{\substack{p=0 \\ p+q < r+s}}^r \sum_{q=0}^s \frac{A_{p,q}^{(j,k)}(z_1, z_2; \alpha, \beta) \alpha_{pq}^{r-p} \beta_{pq}^{s-q}}{(r-p)! (s-q)!} \\ &= \frac{z_1^{r-j} z_2^{s-k}}{(r-j)! (s-k)!} - \sum_{\substack{p=j \\ p+q < r+s}}^r \sum_{q=k}^s \frac{A_{p-j, q-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta)) \alpha_{pq}^{r-p} \beta_{pq}^{s-q}}{(r-p)! (s-q)!} \\ &= \frac{z_1^{r-j} z_2^{s-k}}{(r-j)! (s-k)!} - \sum_{\substack{p=0 \\ p+q < r-j+s-k}}^{r-j} \sum_{q=0}^{s-k} \frac{A_{p,q}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta)) \alpha_{p+j, q+k}^{r-j-p} \beta_{p+j, q+k}^{s-k-q}}{(r-j-p)! (s-k-q)!} \\ &= A_{r-j, s-k}(z_1, z_2; R_{jk}(\alpha), R_{jk}(\beta)), \end{aligned}$$

and this completes the proof.

Lemma 2 and the expansion (2.1) provide a useful expression for the polynomials $A_{m,n}$. Replacing α and β by γ and δ , respectively,

and applying (2.1) to the polynomial $A_{r,s}(z_1, z_2; \alpha, \beta)$, we have

$$(2.5) \quad \begin{aligned} & A_{r,s}(z_1, z_2; \alpha, \beta) \\ &= \sum_{p=0}^r \sum_{q=0}^s A_{r,s}^{(p,q)}(\gamma_{pq}, \delta_{pq}; \alpha, \beta) A_{p,q}(z_1, z_2; \gamma, \delta) \\ &= \sum_{p=0}^r \sum_{q=0}^s A_{p,q}(z_1, z_2; \gamma, \delta) A_{r-p,s-q}(\gamma_{pq}, \delta_{pq}; R_{pq}(\alpha), R_{pq}(\beta)). \end{aligned}$$

If each of γ and δ is the zero matrix, it is easy to see that

$$A_{p,q}(z_1, z_2; \gamma, \delta) = \frac{z_1^p z_2^q}{p! q!}.$$

In this case (2.5) yields

$$(2.6) \quad A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^r \sum_{q=0}^s A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) \frac{z_1^p z_2^q}{p! q!}.$$

Let m and n be integers such that $0 \leq m \leq r$, $0 \leq n \leq s$, and $m + n > 0$. In (2.5) choose

$$\gamma_{pq} = \begin{cases} 0, & \text{if } p \geq m \text{ and } q \geq n \\ \alpha_{pq}, & \text{otherwise} \end{cases}$$

and

$$\delta_{pq} = \begin{cases} 0, & \text{if } p \geq m \text{ and } q \geq n \\ \beta_{pq}, & \text{otherwise.} \end{cases}$$

In view of (2.3) we have

$$(2.7) \quad \begin{aligned} & A_{r,s}(z_1, z_2; \alpha, \beta) \\ &= \sum_{p=m}^r \sum_{q=n}^s A_{p,q}(z_1, z_2; \gamma, \delta) A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)). \end{aligned}$$

More generally, we define the operator P_{jk} as follows. If $j + k > 0$, then $P_{jk}(\alpha)$ is the matrix (a_{pq}) , where

$$a_{pq} = \begin{cases} 0, & \text{if } p \geq j \text{ and } q \geq k \\ \alpha_{pq}, & \text{otherwise.} \end{cases}$$

Then (2.7) becomes

$$(2.8) \quad \begin{aligned} & A_{r,s}(z_1, z_2; \alpha, \beta) \\ &= \sum_{p=m}^r \sum_{q=n}^s A_{p,q}(z_1, z_2; P_{mn}(\alpha), P_{mn}(\beta)) A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)). \end{aligned}$$

Equation (2.8) may be regarded as a separation of variables formula, in the following sense. If $p \geq m$ and $q \geq n$, then $R_{pq}(\alpha)$ depends on the parameters α_{jk} , where $j \geq m$ and $k \geq n$, and $P_{mn}(\alpha)$ depends

on the parameters α_{jk} , where $j < m$ or $k < n$. The usefulness of (2.8) is seen in the next lemma.

LEMMA 3. *If $0 \leq m \leq r$ and $0 \leq n \leq s$, then*

$$(2.9) \quad H_{r,s} \geq H_{m,n} H_{r-m,s-n} .$$

Proof. If $m + n = 0$, the result is trivial. Suppose $m + n > 0$ and choose matrices α and β , whose entries lie on $|z| = 1$, such that

$$H_{m,n} = | A_{m,n}(0, 0; P_{m,n}(\alpha), P_{m,n}(\beta)) |$$

and

$$H_{r-m,s-n} = | A_{r-m,s-n}(0, 0; R_{m,n}(\alpha), R_{m,n}(\beta)) | .$$

For each complex number λ , define the matrices $\gamma = \gamma(\lambda)$ and $\delta = \delta(\lambda)$ by

$$\gamma_{pq} = \begin{cases} \alpha_{pq}, & \text{if } p \geq m \text{ and } q \geq n \\ \lambda \alpha_{pq}, & \text{otherwise} \end{cases}$$

and

$$\delta_{pq} = \begin{cases} \beta_{pq}, & \text{if } p \geq m \text{ and } q \geq n \\ \lambda \beta_{pq}, & \text{otherwise .} \end{cases}$$

By (2.8) and (2.2),

$$\begin{aligned} & A_{r,s}(0, 0; \gamma, \delta) \\ &= \sum_{p=m}^r \sum_{q=n}^s A_{p,q}(0, 0; P_{m,n}(\gamma), P_{m,n}(\delta)) A_{r-p,s-q}(0, 0; R_{pq}(\gamma), R_{pq}(\delta)) \\ &= \sum_{p=m}^r \sum_{q=n}^s \lambda^{p+q} A_{p,q}(0, 0; P_{m,n}(\alpha), P_{m,n}(\beta)) A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) \\ &= \lambda^{m+n} Q(\lambda) , \end{aligned}$$

where $Q(\lambda)$ is a polynomial in λ . Since

$$H_{r,s} \geq \max_{|\lambda|=1} | A_{r,s}(0, 0; \gamma, \delta) | = \max_{|\lambda|=1} | Q(\lambda) | \geq | Q(0) |$$

and

$$\begin{aligned} | Q(0) | &= | A_{m,n}(0, 0; P_{m,n}(\alpha), P_{m,n}(\beta)) | | A_{r-m,s-n}(0, 0; R_{m,n}(\alpha), R_{m,n}(\beta)) | \\ &= H_{m,n} H_{r-m,s-n} , \end{aligned}$$

we have

$$H_{r,s} \geq H_{m,n} H_{r-m,s-n} .$$

LEMMA 4. *There is an infinite subsequence $S = \{(m_j, n_j): j = 1, 2, 3, \dots\}$ such that*

$$(i) \quad H = \lim_{j \rightarrow \infty} H_{m_j, n_j}^{1/(m_j + n_j)}$$

and

$$(ii) \quad H_{m_j, n_j}^{1/(m_j + n_j)} \geq H_{p, q}^{1/(p+q)}$$

for all p and q such that $p + q \leq m_j + n_j$.

Proof. If there is a pair (r, s) such that $H_{r, s}^{1/(r+s)} = H$, then (2.9) implies

$$H \geq H_{j^r, j^s}^{1/j(r+s)} \geq (H_{r, s}^j)^{1/j(r+s)} = H_{r, s}^{1/(r+s)} = H$$

for $j = 1, 2, 3, \dots$. In this case we take $S = \{(j^r, j^s): j = 1, 2, 3, \dots\}$.

Suppose, on the other hand, that $H > H_{r, s}^{1/(r+s)}$ for all r and s . For each positive integer k , let

$$T_k = \max_{p+q=k} H_{p, q}^{1/(p+q)}.$$

Then $T_k < H (1 \leq k < \infty)$ and $\sup_{1 \leq k < \infty} T_k = H$. We can therefore find a subsequence $\{T_{k_j}\}_{j=1}^\infty$ with the properties that

$$\lim_{j \rightarrow \infty} T_{k_j} = H$$

and

$$T_{k_j} > T_n$$

for $n < k_j$. For each j , choose integers m_j and n_j such that $m_j + n_j = k_j$ and $T_{k_j} = H_{m_j, n_j}^{1/(m_j + n_j)}$, and let $S = \{(m_j, n_j): j = 1, 2, 3, \dots\}$. This completes the proof of the lemma.

COROLLARY 2. $H = \limsup_{m+n \rightarrow \infty} H_{m, n}^{1/(m+n)}$.

LEMMA 5. *For each pair of nonnegative integers (m, n) we have*

$$(2.10) \quad H_{m, n} \leq (2/\log 2)^{m+n}.$$

Proof. The result is trivial if $m + n = 0$. Let N be a positive integer and suppose (2.10) holds whenever $m + n < N$. Let r and s be nonnegative integers such that $r + s = N$. The defining relations (1.4) imply

$$\begin{aligned}
 H_{r,s} &\leq \sum_{\substack{p=0 \\ p+q < r+s}}^r \sum_{\substack{q=0 \\ j+k > 0}}^s \frac{H_{p,q}}{(r-p)!(s-q)!} = \sum_{\substack{j=0 \\ j+k > 0}}^r \sum_{k=0}^s \frac{H_{r-j,s-k}}{j!k!} \\
 &\leq \sum_{\substack{j=0 \\ j+k > 0}}^r \sum_{k=0}^s \frac{(2/\log 2)^{r-j+s-k}}{j!k!} \\
 &= (2/\log 2)^{r+s} \left\{ \sum_{j=0}^r \sum_{k=0}^s \frac{((\log 2)/2)^{j+k}}{j!k!} - 1 \right\} \\
 &< (2/\log 2)^{r+s} \left\{ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{((\log 2)/2)^{j+k}}{j!k!} - 1 \right\} \\
 &= (2/\log 2)^{r+s} \{e^{(2/\log 2)} - 1\} = (2/\log 2)^{r+s}.
 \end{aligned}$$

COROLLARY 3. $H \leq (2/\log 2)$.

Note that this result, together with Corollary 1, implies Džrbašjan’s estimate $\mathscr{W} \geq (\log 2)/2$.

3. Main Results. Let

$$M(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{H_{p,q}} \frac{z_1^p z_2^q}{p!q!}.$$

Note that $M(z_1, z_2)$ is an entire function of exponential type 1 or less. Suppose α and β have entries lying in $|z| \leq 1$. By (2.6),

$$A_{r,s}(z_1, z_2; \alpha, \beta) = \sum_{p=0}^r \sum_{q=0}^s A_{r-p,s-q}(0, 0; R_{p,q}(\alpha), R_{p,q}(\beta)) \frac{z_1^p z_2^q}{p!q!}.$$

Since

$$|A_{r-p,s-q}(0, 0; R_{p,q}(\alpha), R_{p,q}(\beta))| \leq H_{r-p,s-q} \leq H_{r,s}/H_{p,q},$$

it follows that the coefficients of $A_{r,s}$ are bounded by the respective coefficients of $H_{r,s}M(z_1, z_2)$; i.e., $A_{r,s}$ is majorized by $H_{r,s}M(z_1, z_2)$. In particular,

$$(3.1) \quad |A_{r,s}(z_1, z_2; \alpha, \beta)| \leq H_{r,s}M(|z_1|, |z_2|).$$

We are now ready to prove Theorem 1.

Suppose f is an entire function, with $\tau(f) < 1/H$, and suppose α and β are matrices whose entries lie in $|z| \leq 1$. In order to justify the expansion (2.1) we show that the series

$$(3.2) \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} |f^{(r,s)}(0, 0)| \sum_{p=0}^r \sum_{q=0}^s \frac{|A_{p,q}(z_1, z_2; \alpha, \beta)|}{(r-p)!(s-q)!}$$

is convergent. Equation (3.1) implies

$$|A_{p,q}(z_1, z_2; \alpha, \beta)| \leq H_{p,q} M(|z_1|, |z_2|) \leq H_{r,s} M(|z_1|, |z_2|) / H_{r-p, s-q};$$

therefore

$$\begin{aligned} & \sum_{p=0}^r \sum_{q=0}^s \frac{|A_{p,q}(z_1, z_2; \alpha, \beta)|}{(r-p)!(s-q)!} \\ & \leq H_{r,s} M(|z_1|, |z_2|) \sum_{p=0}^r \sum_{q=0}^s \frac{1}{H_{r-p, s-q} (r-p)!(s-q)!} \\ & < H_{r,s} M(|z_1|, |z_2|) M(1, 1). \end{aligned}$$

The series (3.2) is therefore convergent provided that

$$(3.3) \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} |f^{(r,s)}(0, 0)| H_{r,s}$$

converges. Choose $\varepsilon > 0$ such that $\tau(f) + \varepsilon < 1/H$ and let N be a positive integer such that $r + s \geq N$ implies

$$|f^{(r,s)}(0, 0)|^{1/(r+s)} < \tau(f) + \varepsilon.$$

Then

$$\sum_{r+s \geq N} \sum |f^{(r,s)}(0, 0)| H_{r,s} \leq \sum_{r+s \geq N} \sum [H(\tau(f) + \varepsilon)]^{r+s}.$$

Let $\rho = H(\tau(f) + \varepsilon)$ and $K = \sum \sum_{r+s < N} |f^{(r,s)}(0, 0)| H_{r,s}$. Then (3.3) is less than

$$K + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \rho^{r+s} = K + \frac{1}{(1-\rho)^2}$$

and the convergence of (3.2) follows.

Proof of Theorem 2. Let $S = \{(m_j, n_j): j = 1, 2, 3, \dots\}$ be an infinite sequence such that

$$H = \lim_{j \rightarrow \infty} H_{m_j, n_j}^{1/(m_j + n_j)}$$

and

$$H_{m_j, n_j}^{1/(m_j + n_j)} \geq H_{p,q}^{1/(p+q)}$$

for all p and q such that $p + q \leq m_j + n_j$. For each $(r, s) \in S$, let $\alpha = \alpha(r, s)$ and $\beta = \beta(r, s)$ be matrices with entries on $|z| = 1$ such that

$$|A_{r,s}(0, 0; \alpha, \beta)| = H_{r,s}.$$

Let

$$P_{r,s}(z_1, z_2) = \frac{A_{r,s}(z_1, z_2; \alpha, \beta)}{A_{r,s}(0, 0; \alpha, \beta)}$$

and

$$Q_{r,s}(z_1, z_2) = P_{r,s}\left(\frac{z_1 H_{r,s}^{1/(r+s)}}{H}, \frac{z_2 H_{r,s}^{1/(r+s)}}{H}\right).$$

Then $Q_{r,s}(0, 0) = P_{r,s}(0, 0) = 1$, and

$$(3.4) \quad Q_{r,s}^{(j,k)}\left(\frac{H\alpha_{jk}}{H_{r,s}^{1/(r+s)}}, \frac{H\beta_{jk}}{H_{r,s}^{1/(r+s)}}\right) = 0 \quad (j < r, k < s),$$

Moreover, (2.6) implies

$$Q_{r,s}(z_1, z_2) = \sum_{p=0}^r \sum_{q=0}^s \frac{A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) H_{r,s}^{(p+q)/(r+s)}}{A_{r,s}(0, 0; \alpha, \beta) H^{p+q}} \frac{z_1^p z_2^q}{p! q!}$$

and

$$\begin{aligned} & \left| \frac{A_{r-p,s-q}(0, 0; R_{pq}(\alpha), R_{pq}(\beta)) H_{r,s}^{(p+q)/(r+s)}}{A_{r,s}(0, 0; \alpha, \beta) H^{p+q}} \right| \\ & \leq \frac{H_{r-p,s-q} H_{r,s}^{(p+q)/(r+s)}}{H_{r,s} H^{p+q}} \leq \frac{H_{r,s}^{(r-p+s-q)/(r+s)} H_{r,s}^{(p+q)/(r+s)}}{H_{r,s} H^{p+q}} = \frac{1}{H^{p+q}}, \end{aligned}$$

since $(r, s) \in S$. Therefore $Q_{r,s}$ is majorized by

$$\varphi(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{H^{p+q}} \frac{z_1^p z_2^q}{p! q!};$$

$\varphi(z_1, z_2)$ is an entire function of exponential type $1/H$. The sequence $\{Q_{m_j, n_j}\}$ is therefore uniformly bounded on compact sets. Extract a uniformly convergent subsequence from $\{Q_{m_j, n_j}\}$ and let F denote the limit function. Then F is entire, $F(0, 0) = 1$, and $\tau(F) \leq 1/H$. Since $F^{(j,k)}$ is the uniform limit of a subsequence of $\{Q_{m_j, n_j}^{(j,k)}\}$, then (3.4) implies that $F^{(j,k)}$ has a zero in $\{|z_1| = 1, |z_2| = 1\}$. The expansion (1.5) implies that F has exponential type exactly $1/H$, and this completes the proof.

4. The Whittaker Constants W and \mathscr{W} . We have already seen that $\mathscr{W} < W$. The following result provides a precise relationship between \mathscr{W} and W , and a determination of W different from [3] and [1].

THEOREM 3. $\limsup_{m+n \rightarrow \infty} H_{m,n}^{1/(m+n)} = 1/\mathscr{W},$
 $\liminf_{m+n \rightarrow \infty} H_{m,n}^{1/(m+n)} = 1/W.$

Proof. The first equation is a consequence of Corollary 1 and Corollary 2. To prove the second, we require the use of the Gončarov polynomials $G_n(z; z_0, \dots, z_{n-1})$ and the sequence

$$H_n = \max |G_n(0; z_0, \dots, z_{n-1})|.$$

If m is a positive integer, the defining relation (1.4) implies

$$(4.1) \quad A_{m,0}(0, 0; \alpha, \beta) = - \sum_{p=0}^{m-1} \frac{A_{p,0}(0, 0; \alpha, \beta) \alpha_{p,0}^{m-p}}{(m-p)!}.$$

In comparing (4.1) with (1.1), one sees that

$$A_{m,0}(0, 0; \alpha, \beta) = G_m(0; \alpha_{0,0}, \alpha_{1,0}, \dots, \alpha_{m-1,0}).$$

It follows that $H_{m,0} = H_m$ and, similarly, $H_{0,m} = H_m$. By Lemma 3 and (1.2), we have

$$\begin{aligned} H_{m,n}^{1/(m+n)} &\geq (H_{m,0} H_{0,n})^{1/(m+n)} = (H_m H_n)^{1/(m+n)} \\ &> \left(\frac{.16}{W^{m+n}} \right)^{1/(m+n)} = \frac{(.16)^{1/(m+n)}}{W}. \end{aligned}$$

Therefore

$$\liminf_{m+n \rightarrow \infty} H_{m,n}^{1/(m+n)} \geq 1/W.$$

In the other direction,

$$\liminf_{m+n \rightarrow \infty} H_{m,n}^{1/(m+n)} \leq \liminf_{m \rightarrow \infty} H_{m,0}^{1/(m+0)} = \lim_{m \rightarrow \infty} H_m^{1/m} = 1/W,$$

and this completes the proof.

Using (2.10) and the estimate $W < .7378$, one easily obtains an interesting bound on \mathscr{W} . For all r and s , we have

$$H_{r,s} \leq (2/\log 2)^{r+s} < \left(\frac{2}{\log 2} \frac{.7378}{W} \right)^{r+s} < \left(\frac{2.13}{W} \right)^{r+s}$$

and therefore

$$W > \mathscr{W} \geq \frac{W}{2.13}.$$

Some remarks should be made relative to stating the above results in terms of k complex variables, $k > 2$. For $j = 1, 2, \dots, k$, let $\alpha^{(j)} = (\alpha_{n_1, n_2, \dots, n_k}^{(j)})$ denote a k -parameter sequence of complex numbers. The recursion relation corresponding to (1.4) is

$$A_{0,0,\dots,0}(z_1, z_2, \dots, z_k) = 1$$

and

$$\begin{aligned}
 & A_{n_1, n_2, \dots, n_k}(z_1, z_2, \dots, z_k) \\
 &= \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1! \dots n_k!} - \sum_{p_1=0}^{n_1} \dots \sum_{p_k=0}^{n_k} \\
 &\quad \times \frac{A_{p_1, \dots, p_k}(z_1, \dots, z_k) [\alpha_{p_1, \dots, p_k}^{(1)}]^{n_1-p_1} \dots [\alpha_{p_1, \dots, p_k}^{(k)}]^{n_k-p_k}}{(n_1 - p_1)! \dots (n_k - p_k)!}
 \end{aligned}$$

where $p_1 + \dots + p_k < n_1 + \dots + n_k$.

The numbers H_{n_1, \dots, n_k} are also defined in the obvious way and we have

$$\begin{aligned}
 H_{n_1, \dots, n_k} &\geq H_{m_1, \dots, m_k} H_{n_1-m_1, \dots, n_k-m_k} , \\
 H_{n_1, \dots, n_l, 0, \dots, 0} &= H_{n_1, \dots, n_l} .
 \end{aligned}$$

The definition of \mathscr{W}_k , the Whittaker constant in k complex variables, is analogous to the definition of \mathscr{W} in § 1. Apart from notational difficulties, it is a direct extension of the above results to see that

$$\limsup H_{n_1, \dots, n_k}^{1/(n_1+\dots+n_k)} = 1/\mathscr{W}_k$$

and

$$\liminf H_{n_1, \dots, n_k}^{1/(n_1+\dots+n_k)} = 1/W .$$

If $1 \leq l \leq k$, we also have

$$\limsup H_{n_1, \dots, n_l, 0, \dots, 0}^{1/(n_1+\dots+n_l)} = 1/\mathscr{W}_l$$

and

$$\liminf H_{n_1, \dots, n_l, 0, \dots, 0}^{1/(n_1+\dots+n_l)} = 1/W ,$$

and it follows that $\mathscr{W} = \mathscr{W}_2 \geq \mathscr{W}_3 \geq \mathscr{W}_4 \geq \dots$.

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