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In recent papers Chew has found a class of topological rings such that if E is one of them, then a space is E-compact if and only if every E-homomorphism on $C(X, E)$ has a one-point support. We generalize this result to a class of topological near-rings. We also have found some topological near-rings which belong to this class.

Chew [5] proved that for the class of α-topological rings, $\mathscr{E}$, X is E-compact, $E \in \mathscr{E}$, if and only if every E-homomorphism on $C(X, E)$ has a one-point support. He also gave a "determination theorems."

The purpose of this paper is to show that the above results hold true for a class of topological near-rings. Since our arguments are almost identical with those of [5], we shall give only the statement of the results and the necessary definitions with a very brief indication of some proofs.

1. Preliminaries.

DEFINITION 1.1. A near-ring is a triple $\{R, +, \cdot\}$ where R is a nonempty set, each of $+$ and $\cdot$ is an associative binary operation on R such that $\{R, +\}$ is a group (need not be abelian) with identity 0, and the following are satisfied,

(a) for each x, y, and $z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$, and

(b) for each $x \in R$, $0 \cdot x = 0$. See [1].

Note that in [2] this type of near-ring is called D-ring. Examples can be found in [2].

DEFINITION 1.2. A near-ring R that contains more than one element is said to be a division near-ring, or near-field if the set R' of nonzero elements is a multiplicative group; and 1 denotes the unity of R'. See [8] and [9].

DEFINITION 1.3. A topological near-ring is a quadruple $\{R, +, \cdot, \mathscr{T}\}$ such that $\{R, +, \cdot\}$ is a near-ring, and $\mathscr{T}$ is a Hausdorff topology on R such that the mappings

$$f: R \times R \to R \text{ defined by } f((x, y)) = x + y$$

and

$$g: R \times R \to R \text{ defined by } g((x, y)) = x \cdot y$$

are continuous. Compare [1]; and a topological near-field is a topological near-ring $\{R, +, \cdot, \mathscr{T}\}$ such that the mapping

$$h\colon (R', R'|\mathscr{T}) \to (R', R'|\mathscr{T}) \text{ defined by } h(x) = x^{-1}$$

is continuous, where x^{-1} in R' is the inverse of x under $\cdot$. See [13, p. 283].

DEFINITION 1.4. A near-ring homomorphism is a mapping ϕ of a near-ring R into a near-ring R_0 such that

$$\phi(\gamma_1 + \gamma_2) = \phi(\gamma_1) + \phi(\gamma_2)$$
$$\phi(\gamma_1 \cdot \gamma_2) = \phi(\gamma_1) \cdot \phi(\gamma_2)$$

for all γ_1 and γ_2 in R. See [3].

A subset I of a near-ring R is said to be a two-sided ideal, or simply an ideal if $(I, +)$ is a normal subgroup of R such that

(1) $RI \subset I$

(2) $(\gamma_1 + t)\gamma_2 - \gamma_1\gamma_2$ is in I if γ_1 and γ_2 are in R and t is in I. See [3].

Then we can easily show that the kernel of a homomorphism is an ideal. Note that $x \cdot 0 = 0$ for any x in R can be shown by using the left distributive law.

For notation and terminology, basic facts concerning E-compact and E-completely regular spaces, and structures of continuous functions we refer to [10], [11] and [5].

Let $C(X, E)$ be the set of all continuous functions from X into the topological near-ring E, and the operations are defined pointwisely. Then $C(X, E)$ is a near-ring.

Let $H(X, E)$ be the space of all E-homomorphisms on $C(X, E)$ endowed with the relative product topology from $E^{C(X,E)}$, and σ be the parametric (evaluation) map corresponding to $C(X, E)$; i.e., $(\sigma(x))(f) = f(x)$ for each x in X and f in $C(X, E)$. By an E-homomorphism we mean a homomorphism ϕ from $C(X, E)$ into E such that $\phi(e) = e$ for all e in E where e is the constant function, $e[X] = \{e\}$.

We recall Theorems (2.1), (3.8) of [10].

PROPOSITION 1.5. *For any topological space E,*

(a) *A space X is E-completely regular if and only if σ is a homeomorphism.*

(b) *For any E-completely regular space X, $\beta_E X = {}_{ext}cl_P\sigma[X]$, the closure of $\sigma[X]$ in $P = E^{C(X,E)}$*

(c) *A space X is E-compact if and only if σ is a homeomorphism and $\sigma[X]$ is closed in P.*

2. Representation theorems. In this section, E is a topological near-ring.

PROPOSITION 2.1. *For any space X, the space $H(X, E)$ is closed in $E^{C(X,E)}$.*

Proof. See [5, (2.1)].

The next proposition is to give a condition for topological near-rings such that $H(X, E) = cl_P\sigma[X]$.

PROPOSITION 2.2. *Suppose that E is a topological near-ring with the property*

(α) *if ϕ in $H(X, E)$, then the family of zero-sets*

$$\{Z(f): f \in C(X, E), f \in \ker \phi\}$$

has the finite intersection property.
Then $cl_P\sigma[X] = H(X, E)$ for any space X.

We shall call the topological near-ring with the property (α) an α-topological near-ring.

THEOREM 2.3. *Let E be an α-topological near-ring. An E-completely regular space X is E-compact if and only if every E-homomorphism $\phi: C(X, E) \to E$ has a one-point support, $\{p_0\}$, in X.*

More generally, for every E-completely regular space X, E-homomorphisms of $C(X, E)$ into E correspond to the points of $\beta_E X$, the E-compactification of X.

Proof. Combining Prop. (1.5) and Prop. (2.2), we can easily prove the necessity; and use contrapositive to prove the sufficiency. As for the second part, we consider the natural correspondence between $C(X, E)$ and $C(\beta_E X, E)$. See [5, (2.3)].

In Theorem 2.3, we may give an additional condition on E, and then replace E-homomorphisms of $C(X, E)$ into E, by arbitrary homomorphisms of $C(X, E)$ into E. We have

COROLLARY 2.4. *Let E be an α-topological near-ring with the following property,*

(β) *every nonzero endomorphism of E is an automorphism.*
Then an E-completely regular space X is E-compact if and only if

every homomorphism ϕ from $C(X, E)$ into E has a one-point support.

Proof. sufficiency is clear.

Necessity. By assumption, each homomorphism ϕ from $C(X, E)$ into E corresponds to an E-homomorphism $\zeta^{-1} \circ \phi$, where ζ is an automorphism of E defined by $\zeta(e) = \phi(e)$ for each $e \in E$. The result follows immediately.

Now, we shall show the "determination theorems".

COROLLARY 2.5. *For any α-topological near-ring E, two E-compact spaces X and Y are homeomorphic if and only if the near-rings $C(X, E)$ and $C(Y, E)$ are E-isomorphic which means that there is an isomorphism ϕ from $C(X, E)$ onto $C(Y, E)$ with $\phi(e) = e$ for all e in E.*

Proof. The necessity is obvious, and the sufficiency is quite straightforward by combining Prop. (2.3) and the fact the E-isomorphism induces a one-to-one correspondence between $H(X, E)$ and $H(Y, E)$.

COROLLARY 2.6. *Let E be an α-topological near-ring with property (β). Then two E-compact spaces X and Y are homeomorphic if and only if the near-rings $C(X, E)$ and $C(Y, E)$ are isomorphic.*

Proof. Use (2.4).

3. Remarks. In this section, we will see a sufficient condition for a topological near-ring to be an α-topological near-ring, and some examples of α-topological near-rings which satisfy the property (β).

PROPOSITION 3.1. *Suppose that E is a topological near-ring with the following properties:*

(a) *for any $\phi \in H(X, E)$, $\phi(f) = 0$ implies $Z(f) \neq \emptyset$.*

(b) *E has a *-function, i.e., there is a continuous function $x \to x^*$ of E into itself such that $xx^* + yy^* = 0$ implies $x = y = 0$. Then E is an α-topological near-ring.*

Proof is the same as that in [5, (3.1)].

Besides the α-topological rings which, of course, are α-topological near-rings shown in [5, § 3], we have the following α-topological near-rings.

An *ordered* near-field is defined in similar fashion as an ordered field, see [8, (2.1)]. A topological *ordered* near-field is an ordered near-field whose topology is defined in (1.3).

PROPOSITION 3.2. *Any topological* ordered *near-field, E, satisfies properties* (a) *and* (b) *in* (3.1).

Proof. (a) Suppose $f \in C(X, E)$ and $Z(f) = \varnothing$. Then f^{-1} defined by $f^{-1}(x) = [f(x)]^{-1}$ for each x in X is in $C(X, E)$, and $f \cdot f^{-1} = 1$. Hence f cannot be in any proper ideal of $C(X, E)$. If ψ is a nonzero homomorphism from $C(X, E)$ into E, then ker

$$\psi = \{h \in C(X, E) \colon \psi(h) = 0\}$$

is a proper ideal of $C(X, E)$. Hence $f \notin \ker \psi$ which is a contradiction.

(b) Consider the identity mapping for $*$-function, i.e., $x^* = x$. Since E is an ordered near-field $xx^* + yy^* = x^2 + y^2 = 0$ implies $x = y = 0$.

PROPOSITION 3.3. *Let E be a near-field with discrete topology. Then E is an α-topological near-ring.*

Proof. We shall prove this by induction. As the proof in (3.2) (a), if f is in $C(X, E)$ with $Z(f) = \varnothing$, then f does not belong to any kernel of element of $H(X, E)$. Thus, if f_1 in $C(X, E)$ with $\phi(f_1) = 0$, then $Z(f_1) \neq \varnothing$. Assume that for $k = n - 1$, $f_1, \cdots, f_{n-1} \in \ker \phi$, $\bigcap_{i=1}^{n-1} Z(f_i) \neq \varnothing$, but $f_1, \cdots, f_n \in \ker \phi$ with $\bigcap_{i=1}^{n} Z(f_i) = \varnothing$. Let $G_k = \bigcap_{i=1}^{k-1} Z(f_i) \backslash Z(f_k)$, $k = 2, \cdots, n$, and

$$g_k(x) = \begin{cases} [f_k(x)]^{-1} & \text{if } x \in G_k \\ 0 & \text{if } x \notin G_k \, . \end{cases}$$

Then since G_k is both open and closed (as each $Z(f_i)$ is), $g_k \in C(X, E)$. Define $f = f_1 + g_2 f_2 + \cdots + g_n f_n$. Then we can easily show that $Z(f) = \varnothing$. But that $\phi(f) = \phi(f_1) + \phi(g_2) \cdot \phi(f_2) + \cdots + \phi(g_n) \cdot \phi(f_n) = 0$ implies $Z(f) \neq \varnothing$. This is a contradiction. Thus $\bigcap_{i=1}^{n} Z(f_i) \neq \varnothing$.

Finally, since the kernel of a homomorphism of near-ring is an ideal and in a near-field, there is no proper ideal hence each nonzero endomorphism of a near-field is an automorphism. Therefore by (3.2) and (3.3) a topological ordered near-field and a near-field with discrete topology have the properties (α) and (β).

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