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In this note, among other things, we shall establish the following result:

THEOREM. **Let (H, X, π) be an effective transformation group, where X is a compact 3-dim, connected manifold, H is a connected locally compact but non-compact topological group and H acts on X almost periodically. Then H must be a Lie group.**

Moreover, If H is abelian, H must be continuously isomorphic into and dense in a torus group of either dimension two or dimension three. In the case of a three-dimensional torus, X must be a minimal set and itself homeomorphic to a three-dimensional torus. The action of H is the usual group multiplication. In the case of a two-dimensional torus, X must not be a minimal set and it must be homeomorphic to one of the following spaces:

(1) T^3, (2) $S^2 \times T$, (3) $P^2 \times T$, (4) $KL \times T$, (5) S^3,

(6) $(\mathrm{Cl}\,(E_2) \times T) \cup (\mathrm{Cl}\,(MB) \times T)$, intersecting in a circle which, in the right side, is an infinite cycle and, in the left side, a cycle homologous to zero,

where (3), (4), and (6) are non-orientable, and T^m is the m-dim torus, S^m the m-sphere, P^m the real projective m-space, KL the Klein bottle, MB the open Moebius strip and E_m the open m-cell.

If H is not abelian, H must be continuously isomorphic into and dense in one of the following groups:

(b_1) $SO(3)$, (b_2) $SU(2)$, (b_3) $SO(3) \times T$, (b_4) $SO(3) \times SU(2)$, (b_5) $SO(3) \times SO(3)$,

where $SO(3)$ is the special orthogonal group of order 3, $SU(2)$ the special unitary group of order 2 and T the circle group. In the case of the last four groups, X must be a minimal set and homogeneous space, but not a quotient group, of each respective group. In the case of the first group, X must not be a minimal set and must be homeomorphic to one of the following spaces:

(1) $S^2 \times T$, (2) $P^2 \times T$, (3) S^3, (4) P^3

(5) $(P^3 \backslash E_3) \cup (P^3 \backslash E_3)$ intersecting in $\mathrm{Cl}\,(E_3) \backslash E_3$

(6) $S^2 \times I$, identifying $(x, 0)$ with $(-x, 1)$.

where (2) and (6) are non-orientable manifolds and I is the closed unit interval.

The main tool of the proof of this theorem is to associate the given transformation group with a compact transformation group and apply a partially affirmative answer to the Hilbert-Montgomery conjecture. Consequently, in general, we are unable to extend the result beyond dimension three. However, we do extend this result to the case of minimal sets.

2. Definition and proof. We adopt the following standard definition (see [3] and [4]).

DEFINITION. Let (H, Y, π) be a transformation group, where X is compact Hausdorff. This transformation group is called almost periodic if β is an index of uniformity in X implies there exists a syndetic subset A of H such that for $x \in X$, we have $\pi(A, x) \subset \beta(x)$. A subset A in H is called *syndetic* if there is a compact subset K in H such that $A \cdot K = H$. This transformation group is called a *minimal set* if the smallest invariant closed set, under H, in X is X itself.

Proof of the theorem. Since X is compact Hausdorff, it is known (see [5]) that H acts on X almost periodically if and only if the action of H on X is uniformly equicontinuous. Consider $f.\ H \rightarrow X^X$ for each $t \in H$, $f(t) = \{\pi(t, x) \mid x \in X\}$, where X^X is the set of all maps of X into X with its usual cartesian product topology, which is compact Hausdorff. Let $G = \mathrm{Cl}\,(f(H))$. Since the action of H on X is uniformly equicontinous it is easy to see that G, as a group of homeomorphisms on X, is a compact topological group. Define $G \times X \xrightarrow{\pi'} X$ by $\pi'(g, x) = g(x)$, where $g \in G$ and $x \in X$. It is not hard to verify that (G, X, π') is a transformation group and the following diagram is commutative:

$$\begin{tikzcd}
(H, X) \arrow[r, "\pi"] \arrow[d, "f\ \ i"'] & X \arrow[d, "i"'] \\
(G, X) \arrow[r, "\pi'"] & X
\end{tikzcd}$$

where i is the identity map. The group H acts on X effectively; so does G. The group H is connected; so is G. It is known that if a compact, connected group acts on a compact three-dimensional manifold effectively, the group must be a Lie group (e.g., see [8]). Since H is locally compact and F is one-to-one into the Lie group G, H must be a Lie group. It is well known that if the maximal dimension of any orbit, under G, is k, then the dimension of G is less than or equal to $1/2k(k+1)$. If G is abelian, then G is a torus. Suppose $k = 1$, then dim. $G = 1$ and G is the circle group. Since H is connected, but not compact and $f(H)$ is a nontrivial connected subgroup of G, we have $f(H)=G$ and f is onto and open. It follows that f is a topological isomorphism and H itself is a circle group. This is a contradiction to the fact that H is not compact. Hence, $k \neq 1$. If $k=2$, then $2 \leqq \dim. G \leqq 3$ and G is a torus of dimension either 2 or 3. Suppose $\dim. G=3$. Then, from the fact that the maximal dimension of any orbit under G is 2, there exists $x_0 \leqq X$ such that its isotropy subgroup G_{x_0} is of dimension one. Thus G_{x_0} contains a circle group and we may write $G_{x_0}=T \times D$, where T is the circle group and D is a finite group. For $x \in X$, we denote by $m(x)$ the number of components in the isotropy subgroup G_x of x. Let $X_2=\{x \mid x \in X, \dim. G(x)=2\}$. Let $s=\min. m(x)$, $x \in X_2$. It is known that the set $X_{2,s}=\{x \mid x \in X_2, m(x)=s\}$, which is called the union of all principal orbits, is dense in X (see [5]). Since G is connected, it is known that $X_{2,s}$ is connected (see [5]). It is also known that, for each $x \in X$, there exists a neighborhood U_x of x such that for each $y \in U_x$, G_y is conjugate to a subgroup of G_x (e.g., see [6]). Since G is abelian, G_y must be a subgroup of G_x. It follows that $X_{2,s}$ is open and every G_x for $x \in X_{2,s}$ contains a circle group, T'. Since $\bar{X}_{2,s}=X$, we can conclude that for each $x \in X$, G_x contains the same circle group T'. This gives a contradiction to the fact that G acts on X effectively. Hence, if $k=2$, the group G must be a two-dimensional torus. In this case, G can not act on X transitively; it follows that X is not minimal under H. If $k=3$, then $3 \leqq \dim. G \leqq 6$ and G must act on X transitively. Suppose $\dim. G>3$, then for each $x \in X$, $\dim. G_x \geqq 1$. From the fact that G is abelian, it follows that G can not act on X effectively. Hence, if $k=3$, the group G must be a three-dimensional torus and for each $x \in X$, G_x is trivial. Consequently, we can introduce a torus group structure on X, so that the action of the group G on X is just the group multiplication. It follows that the action of H on X is the same group multiplication. In the case that $k=2$, the group G is a two-dimensional torus and the space X is a three-dimensional compact connected manifold. It is known (e.g., see theorem 7, [7]) that X must be homeomorphic to one of the following spaces:

(1) T^3, (2) $S^2 \times T$, (3) $P^2 \times T$, (4) $KL \times T$, (5) S^3,

(6) $(\mathrm{Cl}(E_2) \times T) \cup (\mathrm{Cl}(MB) \times T)$, intersecting in a circle which, in the right side, is an infinite cycle and, in the left side, a cycle homologous to zero.

Now, consider the case that G is not abelian. If $k=1$, then $\dim. G=1$. This can not happen because there are no one-dimensional compact connected, non-abelian, Lie groups. If $k=2$, then $2 \leqq \dim. G \leqq 3$. There are only two three-dimensional compact connected, non-abelian, Lie groups, that is $SU(2)$ and $SO(3)$. There are no two-dimensional compact connected, non-abelian, Lie groups. Suppose $G = SU(2)$. If follows that for each $x \in X$, dim. $G_x \geqq 1$. It is known that any nonnormal subgroup of $SU(2)$ must contain its center C, where $C = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\}$. There are no other discrete, nontrivial, normal subgroup other than C. Since $SU(2)$ is simple, every isotropy subgroup G_x is nonnormal. It follows that $G_x \supset C$ for every $x \in X$ and G can not act on X effectively. A contradiction! Hence, if $k = 2$, G must be $SO(3)$. Since $k = 2$, G can not act on X transitively; it follows that X is not minimal under H. In this case, it is known (e.g., see Theorem 7, [7]) that X must be homeomorphic to one of the following spaces:

(1) $S^2 \times T$, (2) $P^2 \times T$, (3) S^3, (4) P^3,
(5) $(P^3 \backslash E_3) \cup (P^3 \backslash E_3)$ intersecting in $\mathrm{Cl}\,(E_3) \backslash E_3$
(6) $S^2 \times I$, identifying $(x, 0)$ with $(-x, 1)$.

If $k = 3$, then $3 \leqq \dim. G \leqq 6$ and G must act on X transitively. Consequently, X is a minimal set under H. If $\dim. G = 3$, then $SU(2)$ and $SO(3)$ are the only two non-abelian, compact, connected, Lie groups. It is possible, for example, that $SU(2)$ and $SO(3)$ can act on themselves with their group multiplications. If $\dim. G = 4$, then $\dim G_x \geqq 1$ for every $x \in X$ and there is $x \in X$ such that $\dim G_x = 1$. The only possible four-dimensional, non-abelian, compact, connected, Lie groups are $SU(2) \times T$ and its quotient groups by factoring a discrete normal subgroup. Since every nonnormal subgroup of $SU(2)$ contains its center C, it is not hard to see that every nonnormal subgroup of $SU(2) \times T$ contains a nontrivial central subgroup of $SU(2) \times T$ whose projection on $SU(2)$ is C. From the fact that $\dim. G_x \geqq 1$ for every $x \in X$, G_x must contain a nontrivial normal subgroup. This is impossible because (G, X, π') is a transitive and effective transformation group. Thus it is not hard to see now that the only possible case left is the group $(SU(2)/C) \times T$, which is isomorphic to $SO(3) \times T$. Indeed, this is possible; for example, let

$$L = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, -\pi \leqq \theta \leqq \pi \right\}$$

be a circle group in $SO(3)$, and choose $X = (SO(3)/L) \times T$. Let $SO(3) \times T$ act on X by the action induced by the group multiplication in $SO(3) \times T$. It is easy to verify that for every $x \in X$, G_x is conjugate to $L \times \{0\}$ and $SO(3) \times T$ acts on X effectively. If $\dim. G = 5$, then $\dim. G_x \geqq 2$ for every $x \in X$ and there is $x \in X$, such that $\dim. G_x = 2$. The only possible five-dimensional non-abelian, compact, connected, Lie groups are $SU(2) \times T^2$ and its quotient groups by factoring a discrete normal subgroup. Choose a $y \in X$ such that dim. $G_y = 2$. On one hand, since (G, X, π') is transitive and effective, the group G_y can not contain a nontrivial normal subgroup. On the other hand, from the fact that dim. $G_y = 2$, it is not hard to see that G_y must contain a circle group which is contained in the center. A contradiction! Hence it is impossible that dim. $G = 5$ and $k = 3$. If dim. $G = 6$, then dim. $G_x \geqq 3$ for every $x \in X$ and there is $x \in X$, such that dim. $G_x = 3$. The only possible six-dimensional, non-abelian, compact, connected, Lie groups are $SU(2) \times T^3$, $SU(2) \times SU(2)$ and their quotinent groups by factoring a discrete normal subgroup. If $G = SU(2) \times T^3$ or its quotient groups by factoring a discrete normal subgroup, then, from the fact that there is $y \in X$ such that dim. $G_y = 3$, G_y must contain a nontrivial normal subgroup. This gives a contradiction to the fact that (G, X, π') is transitive and effective. The group G also can not be $SU(2) \times SU(2)$. Otherwise any non-normal subgroup must contain a nontrivial central subgroup which is contained in $C \times C$; then G can not act on X effectively. Since every discrete normal subgroup of $SU(2) \times SU(2)$ is in the center, $C \times C$, there are four possibilities left: (1) $SU(2) \times SU(2)/C \times C$, which is isomorphic to $SO(3) \times SO(3)$, (2) $SU(2) \times SU(2)/\{e\} \times C$, which is isomorphic to $SU(2) \times SU(3)$, (3) $SU(2) \times SU(2)/C \times \{e\}$, which is isomorphic to $SO(3) \times SU(2)$, (4) $SU(2) \times SU(2)/C'$, where

$$C' = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right\} .$$

It is easy to see that the groups (2), (3) and (4) are isomorphic. It is easy to construct homogeneous spaces so that the action of these groups on the spaces satisfies the given conditions. The theorem is proved.

Since the proof of this theorem depends on a partially affirmative answer to the Hilbert-Montgomery conjecture on effective compact transformation on a manifold, we are unable to extend this result beyond dimension three. However, using the same method and a known result, which says that an effective and transitive compact transformation group acting on a connected finite-dimensional manifold is a Lie group (e.g., see [6]), we can have the following result:

COROLLARY. *Let (H, X, π) be an effective almost periodic transformation group, where X is a compact, connected, finite dimensional manifold and X is a minimal set under H. Then X must be a homogeneous space of a compact, connected Lie group and the action of each element in H on X must be analytic. Furthermore, if H is locally compact, then H must be a Lie group.*

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