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THE TRANSLATION GROUPS OF *n*-UNIFORM TRANSLATION HJELMSLEV PLANES

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The purpose of this paper is twofold: first, to determine the full translation groups for all n-uniform translation affine Hjelmslev planes for all positive integers n; and second, to prove that all such groups occur as the full translation groups of Pappian Hjelmslev Planes.

1. Introduction. For brevity's sake, we introduce the following three conventions: Hjelmslev plane will be abbreviated to H-plane; we will always mean affine (rather than projective) when we write translation *H*-plane; and throughout the paper, translation group will denote the group of all translations. H. Lüneburg has previously defined translation H-planes [7] and has determined the translation groups of all uniform translation *H*-planes [7, Satz 8.3]. The author has defined a class of finite H-planes called n-uniform H-planes in such a way that the finite uniform H-planes are just the n-uniform H-planes with n = 1 and 2 [3]. In §2, we prove (see Theorem 2.6.) that only certain groups can occur as translation groups of *n*-uniform translation *H*-planes; and in \S 3, we establish the converse. As algebraic corollaries to the geometric theorem of $\S 2$, we obtain results on the additive structures of the finite Desarguesian H-rings. (See Corollary 3.1 and Remark 3.3.). This is possible, because every Desarguesian H-ring coordinatizes a Desarguesian affine H-plane, because every Desarguesian affine H-plane is a translation H-plane, and because every finite Desarguesian H-plane is n-uniform for some n.

In § 3, we quote a result of W. E. Clark and the author on the additive structure of finite commutative Desarguesian *H*-rings; we use this result to show that all groups permitted by Theorem 2.6 do in fact occur as translation groups of Pappian affine *H*-planes. Then every translation group of an *n*-uniform translation *H*-plane *A* is isomorphic to the translation group of a Pappian affine *H*-plane *B*. One may always take *B* to have the same invariants as *A*. One may also always choose *B* so that its associated ordinary affine plane has prime order.

2. The translation groups of finite translation *H*-planes. The reader is referred to P. Dembowski [2] or to the papers in the bibliography for definitions of affine and projective *H*-planes. We will write $P \sim Q$, $g \not\sim h$, etc., to mean the point P is neighbor to Q, the line g is not neighbor to h, etc. Associated with every finite

affine or projective *H*-plane are two invariants denoted by s and t. We may take t to be the number of lines through a point P which are neighbor to the line g where (P, g) is an arbitrary flag of the *H*-plane; then s + t will denote the total number of lines incident with P. It is well known that s/t is the order of the ordinary affine or projective plane associated with the *H*-plane. (See [4] and [7].)

DEFINITION 2.1. Let P be a point of an H-plane π . We define \overline{P} to be the following incidence structure. The points of \overline{P} are the points Q of π such that $Q \sim P$. The lines of \overline{P} are the nonempty point sets $l^P = l \cap \overline{P}$, l being a line of π . Incidence is given by inclusion.

DEFINITION 2.2. We define a 1-uniform affine (projective) H-plane to be a finite ordinary affine (projective) plane. We call a finite affine or projective H-plane n-uniform $(n \ge 2)$ provided that

(a) \overline{P} is an (n-1)-uniform affine *H*-plane for each point *P* in π .

(b) For each \overline{P} , every line l^{P} is the restriction of the same number of lines from π .

The following result is part of [3, Proposition 2.2]. The reader should thoroughly acquaint himself with the content of this proposition as it will be used frequently in the rest of the paper.

PROPOSITION 2.1. Let π be an n-uniform projective or affine *H*-plane. Then π satisfies the following properties:

(1) If r = s/t, then $s = r^n$ and $t = r^{n-1}$.

(2) Distinct intersecting neighbor lines of π meet in r^i points for some integer i such that $1 \leq i \leq n-1$.

(3) The dual of (2) holds in π .

(4) If $P \in h$, the number of lines through P which intersect h in r^i or more points is r^{n-i} for $i = 1, 2, \dots, n$.

(5) The dual of (4) holds in π .

We write " $P(\cong i)Q$ " and read "P is *i*-equivalent to Q" to mean P is joined to Q by exactly r^i lines; we write " $P(\sim i)Q$ " and read "P is at least *i*-equivalent to Q" to mean P is joined to Q by r^i or more lines.

(6) (~i) is an equivalence relation on points for $i=0, 1, \dots, n$.

(7) The following conditions imply $|l \cap k| > 1$: $R, Q \in l$; $R, S \in k$; $R(\cong i)Q$; $Q(\sim i + 1)S$; i is a nonnegative integer < n.

(8) If P is any point of π , the number of points Q of π such that $Q(\sim i)P$ is $r^{2(n-i)}$ for $i = 1, 2, \dots, n$.

In light of Proposition 2.1 (1), an *n*-uniform H-plane may be

thought of as having three invariants r, s, and t. However, s and t are determined by r and n; and thus, we shall write the invariant of an *n*-uniform *H*-plane π to refer to r. Since r=s/t, the invariant of π is the order of the ordinary affine or projective plane associated with π . Next we prove

LEMMA 2.2. Let P, Q, R be points of an n-uniform H-plane which satisfy $P,Q \in g$; $P,R \in h$; $Q,R \in k$. Further suppose $P(\cong i)Q(\cong i)R(\cong i)P$, i < n, and $g \neq h$. Then $h \neq k \neq g$.

Proof. Proposition 2.1 (5) implies the number of points X such that $X \in g$ and $X(\sim i + 1)P$ is r^{n-i-1} . By (7), any line joining R to such an X is neighbor to h, hence not neighbor to g. Then no line joins R to two such X. By (6), the number of lines joining R to each such X is r^i . Then the number of lines joining R to all such X is $r^{n-1} = t$. Thus all lines through R which are neighbor to h meet g in points X which satisfy $X(\sim i + 1)P$. Then $k \not\sim h$, and by symmetry $k \not\sim g$.

To state the next several lemmas, we need some notation and a definition. We will write ${}^{i}P$ to denote $\{Q: Q(\sim n - i)P\}$. Thus ${}^{o}P = \{P\}$ and ${}^{n}P$ is the set of all points of the *H*-plane.

DEFINITION 2.3. A mapping σ defined on the point set of an affine *H*-plane is called a *dilatation* if the following condition is satisfied: $P, Q \in g; (P)\sigma \in h; g \parallel h \text{ imply } (Q)\sigma \in h.$

LEMMA 2.3. Let σ be a dilatation of an n-uniform affine H-plane. Let P, Q, R, T be points such that $Q(\cong j)P(\cong j)R$, $T(\sim j + 1)P$, and $(P)\sigma(\cong i + j)(Q)\sigma$. Then

- (a) $(R)\sigma(\cong i+j)(P)\sigma$, and
- (b) $(T)\sigma(\sim i + j + 1)(P)\sigma \text{ if } i + j < n.$

Proof. Let g be any line through P and Q, h be any line through P which is not neighbor to g. We first prove the lemma for all $R, T \in h$ such that $R(\cong j)P, T(\sim j+1)P$. We have $R, T(\sim j)Q$. Since $h \not\sim g$, Proposition 2.1 (7) implies $R(\cong j)Q$. Let k be any line through R ane Q, m be any line through T and Q. By Lemma 2.2, $|h \cap k| = 1$. Let g', h' be the lines through $(P)\sigma$ parallel respectively to g, h; let k', m' be the lines through $(Q)\sigma$ parallel respectively to k, m. In one form of the definition of affine H-planes (See [7] or [3], not [2].), the following condition is assumed: $|h \cap k| = 1$ and $k ||k' imply |h \cap k'| = 1$. Then also $|h' \cap k'| = 1$. Similarly, $|h' \cap g'| = 1$; and since $|m \cap g| > 1, |m' \cap g'| > 1$. Since $(P)\sigma(\cong i+j)(Q)\sigma$ and $h' \not\sim k'$, Pro-

position 2.1 (7) implies $(R)\sigma(\sim i+j)(P)\sigma$. Since $m' \sim g' \not\sim h'$, the same argument implies $(T)\sigma(\sim i+j)(P)\sigma$. We have $(R)\sigma(\cong i+j)(P)\sigma$, for otherwise the above argument would yield $(Q)\sigma(\sim i+j+1)(P)\sigma$. Next, suppose $(T)\sigma(\cong i+j)(P)\sigma$. Then since $h' \not\sim g'$, $(T)\sigma(\cong i+j)(Q)\sigma$. If i+j < n, Lemma 2.2 implies $m' \not\sim g'$. By the contradiction, we conclude that $(T)\sigma(\sim i+j+1)(P)\sigma$. To see that the conclusions of the lemma hold for points R and T on a line h through P such that $h \sim g$, we apply the above results, replacing g and Q by g^* and Q^* where $Q^* \in g^* \not\sim g$ and $Q^*(\cong j)P$. (The existence of such a point $Q^* \in g^*$ is assured by Proposition 2.1 (5).)

LEMMA 2.4. Let σ be a dilatation of an n-uniform affine H-plane. Let $(P)\sigma(\cong i)(Q)\sigma$ for nonneighbor points P, Q. Then if $k \leq n-i$, $\binom{n-k}{P}\sigma = \binom{n-k-i}{(P)\sigma}$.

Proof. Taking j = 0 in Lemma 2.3 yields $({}^{n}P)\sigma \subset {}^{n-i}((P)\sigma)$ and $({}^{n-1}P)\sigma \subset {}^{n-i-1}((P)\sigma)$. It follows from Proposition 2.1 (8) that for each $k = 0, 1, \dots, n-1$, there exists a point R_k such that $R_k(\cong k)P$. Using induction and Lemma 2.3, we get $({}^{n-k}P)\sigma \subset {}^{n-i-k}((P)\sigma)$ for all $k \leq n - i$. If we can prove that the last containment is equality when k = 0, then another induction proof using Lemma 2.3 will yield the full conclusion of Lemma 2.4. Thus it suffices to prove that ${}^{n-i}((P)\sigma) \subset \text{Image } (\sigma)$.

We let g denote the line joining P and Q; g', the line through $(P)\sigma$ which is parallel to g. Let h' be any line through $(P)\sigma$ not neighbor to g', and let R' be any point of h' satisfying $R'(\sim i)(P)\sigma$. Let k' be any line joining R' to $(Q)\sigma$. Since $(Q)\sigma (\cong i)(P)\sigma$ and $R'(\sim i)(P)\sigma, R'(\sim i)(Q)\sigma.$ Since $h' \not\sim g', R'(\cong i)(Q)\sigma.$ If $R'(\sim i+1)(P)\sigma$, then $k' \sim g'$; hence $k' \not\sim h'$. If $R'(\cong i)(P)\sigma$, Lemma 2.2 implies $k' \not\sim h'$. Then in all cases $|k' \cap h'| = 1$. Let h be the line through P which is parallel to h', k be the line through Q which is parallel to k'. Then $|k \cap h| = 1$. If $\{R\}$ is $k \cap h$, then $(R)\sigma = R'$. To see that Image (σ) contains points R' on lines $h' \sim g'$, repeat the above argument using (in place of g and Q) a line g^* through P such that $g^* \not\sim g$ and a point Q^* on g^* such that $Q^* \not\sim P$. Lemma 2.3 implies $(P)\sigma (\cong i)(Q^*)\sigma$. Since $g^* \not\sim g$, a previous argument implies $(g^*)' \not\sim g'$, hence $(g^*)' \not\sim h'$. (Here $(g^*)'$ denotes the line through $(P)\sigma$ which is parallel to g^* .) Then $n-i((P)\sigma) \subset \text{Image}(\sigma)$, and the proof of the lemma is complete.

Lemmas 2.3 and 2.4 combine to yield

PROPOSITION 2.5. Let σ be a dilatation of an n-uniform affine *H*-plane. Let $P(\cong j)Q$ and $(P)\sigma(\cong i+j)(Q)\sigma$ for some j < n-i.

Then for all $k \leq n - i$, $({}^{n-k}P)\sigma = {}^{n-k-i}((P)\sigma)$.

The reader is referred to [7] or [2] for the definition of translation *H*-planes and for the results on translation *H*-planes which we quote and use below. If π is a set of subgroups (called *components*) of the group $T, J(T, \pi)$ denotes the incidence structure with parallel relation defined as follows: the points are the elements of T; the lines are the right cosets of the components; incidence is given by inclusion; and lines are parallel if and only if they are cosets of the same component of π . If A is any translation *H*-plane and if T^* is the translation group of A, then T^* is abelian and there exist T, π such that $A \cong J(T, \pi)$ and $T^* \cong T$. Every element t^* of T^* may be defined on T by $(x)t^* = x + t$ for all $x \in T$, some fixed $t \in T$. If $J(T, \pi)$ is an affine *H*-plane A and if T is abelian, then A is a translation *H*-plane with translation group isomorphic to T. Finally, we note that the invariant of A must be a prime power, since the ordinary affine plane associated with A is a translation plane.

THEOREM 2.6. Let A be an n-uniform translation H-plane with invariant $r = p^{*}$ and translation group T^{*} . Then there exist nonnegative integers k_{1}, k_{2}, j such that T^{*} is the direct sum of $2xk_{1}$ cyclic subgroups of order p^{j} and of $2xk_{2}$ cyclic subgroups of order p^{j+1} .

Proof. We represent A by $J(T, \pi)$ where $T \cong T^*$. Let ${}^{i}T$ denote the set of all elements of T in ${}^{i}0$. Let $\tau \in {}^{i}T, \tau^*$ denote the translation which adds τ to each element of T. Then all lines connecting 0 and τ are "traces" of τ^* , i.e., are fixed by τ^* . Then if $\beta \in T$, all lines through β parallel to these traces are also traces of τ^* , hence contain $(\beta)\tau^*$. Then $\beta(\sim n - i)(\beta)\tau^*$; and if $\beta \in {}^{i}T, \tau + \beta = (\beta)\tau^* \in {}^{i}T$. Then ${}^{i}T$ is a subgroup of T. Let ${}^{i}\pi$ denote the set of all intersections of ${}^{i}T$ with components of π . Then ${}^{i}0$ is isomorphic to $J({}^{i}T, {}^{i}\pi)$. Since A is n-uniform, ${}^{i}0$ is an *i*-uniform affine H-plane; since ${}^{i}T$ is an abelian group, ${}^{i}0$ is a translation H-plane.

We prove the theorem by induction on n. The 1-uniform translation *H*-planes are just the finite translation planes, and it is well known that such planes have elementary abelian translation groups. Since the order of the translation group of such a plane equals r^2 , the number of points in the plane, the theorem is satisfied with $j = 1 = k_1$ and $k_2 = 0$. Now let A be an *n*-uniform translation *H*-plane with n > 1. By the induction hypothesis, n-1T is the direct sum of $2xk_1$ cyclic subgroups of order p^j and of $2xk_2$ cyclic subgroups of order p^{j+1} for suitable k_1, k_2, j . We may assume $j > 0, k_1 > 0$. Let σ be the dilatation of A defined by $(\beta)\sigma = p\beta$ for all $\beta \in T$. By Lemma 2.4, Image $(\sigma) = i0$ for some i < n. If i = 0, T is elementary abelian. The theorem is then satisfied with j=1, $k_1=n$, $k_2=0$, since the number of points in A is r^{2n} .

Henceforth, we assume i > 0. Since i < n, we may apply the induction assumption to ${}^{i}T$ and conclude that T is a p-group even for $i \neq 0$. If $i \neq 0$, Lemma 2.4 implies that $p({}^{n-1}T) = {}^{i-1}T$. Then ${}^{i-1}T$ is the direct sum of $2xk_1$ cyclic subgroups of order p^{j-1} and $2xk_2$ cyclic subgroups of order p^{j} .

Now $o({}^{i}T) = p^{2x} \cdot o({}^{i-1}T)$, and $o(T) = p^{2x} \cdot o({}^{n-1}T)$. Thus, letting σ^* denote the restriction of σ to ${}^{n-1}T$, we see that Ker (σ^*) and Ker (σ) have the same order. Then T and ${}^{n-1}T$ both have the same number $k = 2x(k_1 + k_2)$ of summands. By counting elements of order p, we see that, in general, no p-group may have fewer summands than any of its subgroups. ${}^{i}T$ and ${}^{i-1}T$ also have k summands unless j = 1. Assume j = 1 so that ${}^{i-1}T$ is the direct sum of $2xk_2$ cyclic subgroups of order p. Applying the induction assumption to ${}^{i}T$ and observing that $o({}^{i}T) = p^{2x} \cdot o({}^{i-1}T)$, we see that either

(2.1) ${}^{i}T$ is the direct sum of $2x(k_{2}+1)$ cyclic subgroups of order p,

or

(2.2) ${}^{i}T$ is the direct sum of $2x(k_{2}-1)$ cyclic subgroups of order p and of 2x cyclic subgroups of order p^{2} .

Assume that (2.2) is satisfied, and apply the induction assumption to ${}^{i+1}T, {}^{i+2}T, \cdots, {}^{n-2}T$. Since ${}^{n-1}T$ has more summands than ${}^{i}T$, there is an integer l such that $0 \leq l < n-1$ and ${}^{i+l+1}T$ is isomorphic to the direct sum of ${}^{i+l}T$ and of 2x cyclic subgroups of order p. Then $({}^{i+l+1}T)\sigma = ({}^{i+l}T)\sigma \neq 0$ which contradicts Lemma 2.4. We conclude that (2.1) is the only possibility for ${}^{i}T$ when j = 1.

If j > 1, applying the induction assumption to ${}^{i}T$, we see that

(2.3) ^{*i*}T must be the direct sum of $2x(k_1 - 1)$ cyclic subgroups of order p^{j-1} and of $2x(k_2 + 1)$ cyclic subgroups of order p^j . Since (2.1) is just a degenerate case of (2.3), we see that (2.3) must be satisfied whenever T is not elemetary abelian. Then if T is not elementary abelian, T must contain a subgroup S which is the direct

sum of $2x(k_1 - 1)$ cyclic subgroups of order p^j and $2x(k_2 + 1)$ cyclic subgroups of order p^{j+1} . Since o(T) = o(S), T = S, and the proof is complete.

We have also proved

LEMMA 2.7. For all $m \leq n$, either ^mT is elementary abelian, or else there exist nonnegative integers j, k_1, k_2, x such that

(a) ${}^{m-1}T$ is the direct sum of $2xk_1$ cyclic subgroups of order p^j and of $2xk_2$ cyclic subgroups of order p^{j+1} ; (b) ^mT is the direct sum of $2x(k_1 - 1)$ cyclic subgroups of order p^j and of $2x (k_2 + 1)$ cyclic subgroups of order p^{j+1} .

We now use Lemma 2.7 to obtain the following improvement of Theorem 2.6.

THEOREM 2.6A. Let $A = J(T, \pi)$ be an n-uniform translation H-plane with invariant $r = p^{*}$ and translation group isomorphic to T. Then there exist integers l, k with $0 \leq l < k$ and subgroups C_{i} of T which satisfy the following conditions:

(a) $T = C_1 \bigoplus \cdots \bigoplus C_k$;

(b) for $i \leq l$, C_i is the direct sum of 2x cyclic subgroups of order p^{j+1} ;

(c) for i > l, C_i is the direct sum of 2x cyclic subgroups of order p^j ;

(d) for $i \leq n = kj + l$,

$${}^{i}0 = p^{q+1} \cdot (C_1 \oplus \cdots \oplus C_e) \oplus p^q \cdot (C_{e+1} \oplus \cdots \oplus C_k)$$

where q, e are given by n - i = kq + e, $0 \leq e < k$.

Proof. By Theorem 2.6, we have that T is the direct sum of $2xk_1$ cyclic subgroups of order p^j and $2xk_2$ cyclic subgroups of order p^{j+1} . Set $k = k_1 + k_2$. Using Lemma 2.7 and Proposition 2.1 (8), it is easy to see that for $m \leq k$,

$${}^{m}T = D_{\scriptscriptstyle 1} \oplus \cdots \oplus D_{\scriptscriptstyle m}$$

where each D_i is the direct sum of 2x cyclic subgroups of order p. By Lemma 2.4, there exists an integer c such that $p^bT = {}^{b-c}T$ for all $b \ge c$. Clearly, c = k. Assume that for some m with $0 \le m < \min(k, n - k)$, there exist subgroups E_i of T satisfying

$$(2.4) \quad {}^{m+k}T = E_1 \bigoplus \cdots \bigoplus E_m \bigoplus D_{m+1} \bigoplus \cdots \bigoplus D_k;$$

$$(2.5) \quad pE_i = D_i \text{ for } 1 \leq i \leq m;$$

(2.6) each E_i is the direct sum of 2x cyclic groups of order p^2 . Certainly the above requirements are satisfied for m=0. Let $\{d_i: 1 \leq i \leq 2x\}$ be a basis for D_{m+1} . Since $p(^{m+k+1}T) = {}^{m+1}T \supset D_{m+1}$, there exist $e_i \in {}^{m+k+1}T$ satisfying $pe_i = d_i$. Let E_{m+1} be the group generated by $\{e_i\}$. Suppose e is an element of

$$E_{m+1}\cap (E_1\oplus\cdots\oplus E_m\oplus D_{m+2}\oplus\cdots\oplus D_k)$$
 .

Then by (2.5),

$$pe \in D_{m+1} \cap (D_1 \bigoplus \cdots \bigoplus D_m)$$
.

Then pe = 0; hence

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 $e \in D_{m+1} \cap (D_1 \oplus \cdots \oplus D_m \oplus D_{m+2} \oplus \cdots \oplus D_k)$.

Then e = 0. We have proved that the sum

$$E = E_1 + \cdots + E_{m+1} + D_{m+2} + \cdots + D_k$$

is direct. We know $E \subset {}^{m+k+1}T$, and Proposition 2.1 (8) implies $E = {}^{m+k+1}T$. It is now easy to see that (2.4) - (2.6) are all satisfied if we substitute m + 1 for m.

Proceeding in this manner, we eventually obtain

$$T = F_1 \oplus \cdots \oplus F_k$$

where F_i is the direct sum of 2x cyclic groups of order p^{j+1} if $i \leq k_2$ and of 2x cyclic groups of order p^j when $i > k_2$. The result now follows from setting $l = k_2$, $C_i = F_{l+1-i}$ for $1 \leq i \leq l$ and $C_i = F_{k+l+1-i}$ for $l < i \leq k$. We may assure l < k by changing the value of j if necessary.

3. The translation groups of finite Desarguesian affine H-planes. The reader is referred to Klingenberg [5], [6] or Dembowski [2] for the definition of Desarguesian and Pappian affine H-planes as well as for all the results on such planes stated below. We do repeat the following definition.

DEFINITION 3.1. A Desarguesian H-ring (henceforth abbreviated to H-ring) is an associative ring with identity which satisfies the following three conditions:

(a) Every divisor of zero is a two-sided divisor of zero, and the set N of divisors of zero is an ideal.

(b) Every nondivisor of zero has an inverse.

(c) If $n, m \in N$, then there is an $h \in H$ such that nh = m or n = mh; and there is a $k \in H$ such that kn = m or n = km.

If H denotes an H-ring, then Klingenberg defined [6] an incidence structure $\sum_{p} (H)$ as follows: points are left "homogeneous triples" of elements of H; lines are right "homogeneous triples"; a point and line are incident if and only if the inner products of their corresponding triples are zero. Klingenberg proved [6, S 28, S 29, proof of S 29] that $\sum_{p} (H)$ is a projective H-plane whose affine H-planes are all isomorphic Desarguesian affine H-planes with translation groups isomorphic to $H^+ \bigoplus H^+$. The affine H-planes belonging to $\sum_{p} (H)$ are themselves coordinatizable (in an affine manner) by the ring H and Klingenberg denotes such an affine H-plane by $\sum_{a} (H)$. Call a projective H-plane P Desarguesian if and only if P is isomorphic to $\sum_{p} (H)$ for some H-ring H.

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By definition, all affine Desarguesian *H*-planes are translation *H*-planes. The author has proved [3, Theorem 5.4] that all finite $\sum_{p} (H)$ and hence also all finite $\sum_{a} (H)$ are *n*-uniform for various *n*. Let $\sum_{a} (H)$ or $\sum_{p} (H)$ be *n*-uniform with invariants *r*, *s*, *t*. Then $o(H) = s = r^{n}$ and $o(N) = t = r^{n-1}$ (See [3, Lemma 5.1].) It is clear from [3, Theorem 5.3 and Lemma 5.2 (1)] that $o(N^{i}) = r^{n-i}$ for $1 \leq i \leq n$. In particular, *N* is nilpotent of degree *n*. We are now in a position to state and prove the following algebraic corollary to Theorem 2.6.

COROLLARY 3.1. Let H be a finite H-ring with radical N. Let r^* denote o(H/N). Then r^* is a prime power p^* . H^+ is the direct sum of xk_1 cyclic subgroups of order p^j and of xk_2 cyclic subgroups of order p^{j+1} for some nonnegative integers k_1, k_2, j .

Proof. Since H is a finite H-ring, $\sum_{a} (H)$ is an n-uniform translation H-plane. Since $r^* = o(H/N)$, r^* is the invariant of $\sum_{a} (H)$; hence r^* is a prime power. The result now follows from Theorem 2.6 and the previous observation that the translation group of $\sum_{a} (H)$ is isomorphic to $H^+ \bigoplus H^+$.

We remark that W. E. Clark and the author [1] have given an algebraic proof of Corollary 3.1. Nevertheless, it is interesting that the corollary should be an immediate consequence of a geometric theorem.

LEMMA 3.2. Let H be a finite H-ring with radical N, $\sum_{a} (H)$ be n-uniform. Then for each point (c, d) of $\sum_{a} (H)$, one has ${}^{i}(c, d) = \{(c + a, d + b): a, b \in N^{n-i}\}, 0 \leq i < n$.

Proof. Let $a \in N^{n-i} - N^{n-i+1}$, $b \in N^{n-j} - N^{n-j+1}$. We assume $i \ge j$. Let [x, y] denote the line whose incident points are $\{(tx, ty): t \in H\}$. The lines through (0, 0) are the lines of the form [x, y]. (See [6, S 23]. Note that [x, y] is a line if and only if not both $x, y \in N$.) Let [x, y] be a line through (a, b). Then there exists $t_0 \in H$ such that $a = t_0 x$, $b = t_0 y$; hence $x \in H - N$ and $t_0 \in N^{n-i}$. Let $u \in H - N$, $w \in N^i$, $v = ux^{-1}y + w$. Then [u, v] contains (a, b). There are $(s-t)r^{n-i}$ satisfactory pairs u, v; and, since [u, v] = [u', v'] if and only if u' = zu, v' = zv for a unit z, these must give rise to at least r^{n-i} distinct lines. Then $(a, b) \in {}^i(0, 0)$. Similarly, if $j \ge i$, $(a, b) \in {}^j(0, 0)$. Let X = $\{(a, b): a, b \in N^{n-i}\}$. Then $X \subset {}^i(0, 0)$. Since $|X| = r^{2i} = |{}^i(0, 0)|$, ${}^i(0, 0) = X$. This yields the result when (c, d) = (0, 0). To obtain the full result, one merely considers the translation $\tau(c, d)$ which maps each point (x, y) to (c + x, d + y). REMARK 3.3. Let $H, N, r^* = p^x, k_1, k_2$ be as in Corollary 3.1. Set $k = k_1 + k_2$. Let *i* be any nonnegative integer less than *n* where *n* satisfies $N^{n-1} \neq N^n = 0$. Let *q*, *r* be the nonnegative integers which satisfy i = kq + r and r < k. Then $(N^{n-i})^+$ is the direct sum of x(k - r) cyclic subgroups of order p^q and of xr cyclic subgroups of order p^{q+1} .

Proof. Let $T = H^+ \bigoplus H^+$. Let π be the set of lines [x, y]. It is clear from [6, S23] that $\sum_{a} (H) \cong J(T, \pi)$. By Lemma 3.2, $N^{n-i} \times N^{n-i} = {}^{i}(0, 0) = {}^{i}T$. Then the conclusion follows from Lemma 2.7 and Corollary 3.1.

In [1], W. E. Clark and the author prove the following result:

PROPOSITION 3.4. Let there be given a prime integer p and nonnegative integers x, k_1, k_2, j , such that x > 0 and $k_1j + k_2(j + 1) > 0$. Then there exists a commutative H-ring H with radical N such that $o(H/N) = p^x$ and so that H^+ is the direct sum of xk_1 cyclic subgroups of order p^j and of xk_2 cyclic subgroups of order p^{j+1} .

Klingenberg proves (See [5] or [2].) that if H is a commutative H-ring, then $\sum_{a} (H)$ is Pappian. We then obtain the following strong converse to Theorem 2.6 as an immediate corollary to Proposition 3.4.

COROLLARY 3.5. Let $r = p^x$, k_1 , k_2 , j be given with $k_1 j + k_2 (j+1) > 0$, x > 0. Then there exists a Pappian affine H-plane with invariant r whose translation group is the direct product of $2xk_1$ cyclic subgroups of order p^j and $2xk_2$ cyclic subgroups of order p^{j+1} .

Corollary 3.5 says that all translation groups of *n*-uniform translation *H*-planes can be obtained as translation groups of Pappian *H*-planes. Actually it says somewhat more: namely, if *T* is the translation group of an *n*-uniform translation *H*-plane *A* whose invariant is p^x , then *T* can be obtained as the translation group of a Pappian affine *H*-plane *B* whose invariant is p^y where *y* is any positive integer such that $y | xk_1$ and $y | xk_2$. In particular, one can always take y = x so that *A* and *B* will have the same invariant. Also all translation groups can be obtained as the translation groups of Pappian affine *H*-planes whose associated affine planes are of prime order.

References

1. W. E. Clark and D. A. Drake, Finite chain rings, (To appear)

2. P. Dembowski, *Finite Geometries*, Ergebnisse der Mathematik 44, Springer, Berlin-Heidelberg-New York, 1968.

3. D. A. Drake, On n-uniform Hjelmslev planes, J. Combinatorial Theory 9 (1970), 267-288.

4. E. Kleinfeld, Finite Hjelmslev planes, Illinois J. Math. 3 (1959), 403-407.

5. W. Klingenberg, Projektive und a fine Ebenen mit Nachbarelementen, Math. Z. 60 (1954), 384-406.

6. _____, Desarguessche Ebenen mit Nachbarelementen, Abh. Math. Sem. Univ. Hamburg, **20** (1955), 97-111.

7. H. Lüneburg, Affine Hjelmslev-Ebenen mit transitiver Translationsgruppe, Math. Z. 79 (1962), 260-288.

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