CONVOLUTIONS OF ARITHMETIC FUNCTIONS OVER COHESIVE BASIC SEQUENCES

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Necessary and sufficient conditions are given for the generalized convolution of two arithmetic functions which are units with respect to a basic sequence $\mathscr{B}$ to be again a unit on $\mathscr{B}$. These conditions are then investigated from function-theoretic and combinatorial points of view.

1. **Introduction.** The relation between the multiplicative properties of an arithmetic function and the underlying basic sequence has already been studied in some detail (see [1]). We will confine ourselves here to the consideration of a certain class of basic sequences, and an investigation of the properties of convolutions of arithmetic functions over these basic sequences. The definitions and preliminary results used here may be found in [1], but we will repeat them as a matter of convenience.

A *basic sequence* $\mathscr{B}$ is a set of pairs (a, b) of positive integers with the properties

(i) $(1, k) \in \mathscr{B}$ $(k = 1, 2, \cdots)$,
(ii) $(a, b) \in \mathscr{B}$ if and only if $(b, a) \in \mathscr{B}$,
(iii) $(a, bc) \in \mathscr{B}$ if and only if $(a, b) \in \mathscr{B}$ and $(a, c) \in \mathscr{B}$.

Some examples of basic sequences are

$$\mathscr{S} = \bigcup_{k=1}^{\infty} S_k, \text{ where } S_k = \{(1, k), (k, 1)\},$$

$\mathscr{M} = \{(a, b) \,|\, a \text{ and } b \text{ are relatively prime positive integers}\}$,
$\mathscr{L} = \{(a, b) \,|\, a \text{ and } b \text{ are any positive integers}\}$,
$\mathscr{T}_p = \mathscr{S} \cup \{(p^a, p^b) \,|\, a \text{ and } b \text{ are any positive integers}\}$,

where p is a prime.

A pair (a, b) of positive integers is called a *primitive pair* if both a and b are primes. If $a \neq b$, the pair is a *type I* primitive pair; if $a = b$, the pair is a *type II* primitive pair. We see that a basic sequence is completely determined by the primitive pairs it contains. If Φ is any set of pairs (primitive or not) of positive integers, the basic sequence *generated* by Φ is defined to be

$$\Gamma[\Phi] = \bigcap \mathscr{D},$$

where the intersection is taken over all basic sequences $\mathscr{D}$ which contain Φ. Thus, a basic sequence is generated by its subset of

primitive pairs. An *arithmetic function* is a real-valued function defined on the set of natural numbers. Let f be an arithmetic function and let m and n be natural numbers. We define

$$\alpha_f(m, n) = \begin{cases} \dfrac{f(m)f(n) - f(mn)}{|f(m)f(n)| + |f(mn)|} & \text{if } |f(m)f(n)| + |f(mn)| > 0 , \\ 0 & \text{if } f(m)f(n) = f(mn) = 0 . \end{cases}$$

Let $\mathscr{B}$ be a basic sequence. We say that the *index of multiplicativity* of f with respect to $\mathscr{B}$ exists, and has the value $I(f, \mathscr{B})$, provided

$$\lim_{k \to \infty} \alpha_f(m_k, n_k) = I(f, \mathscr{B})$$

for every collection of pairs $\{(m_k, n_k)\}_{k=1}^{\infty}$ in $\mathscr{B}$ for which $\lim_{k \to \infty} m_k n_k = \infty$.

f is said to be *multiplicative with respect to* $\mathscr{B}$ if $f(m)f(n) = f(mn)$ for every pair (m, n) in $\mathscr{B}$. Functions which are multiplicative with respect to $\mathscr{M}$ are multiplicative in the usual sense. For f to be multiplicative with respect to $\mathscr{B}$, it is clearly necessary that $I(f, \mathscr{B}) = 0$.

If $\mathscr{B}$ is a basic sequence and f is an arithmetic function such that

(1) $I(f, \mathscr{B}) = 0$,

(2)   there exists an integer N such that $f(k) \neq 0$ and $f(k)$ does not change sign for $k \geq N$,

then f is called a *unit* with respect to $\mathscr{B}$. A unit is *positive* or *negative* depending on the sign of $f(k)$ for large values of k.

The present investigation was motivated by an attempt to solve the following problem: Characterize those basic sequences $\mathscr{B}$ for which, given any pair of units f, g on $\mathscr{B}$ and any basic sequence $\mathscr{C}$ containing $\mathscr{B}$, the convolution $f \circ_{\mathscr{C}} g$ is again a unit on $\mathscr{B}$, where

$$(f \circ_{\mathscr{C}} g)(n) = \sum_{(d, n/d) \in \mathscr{C}} f(d)g(n/d) .$$

In view of the following lemma we need consider only basic sequences $\mathscr{B}$ which are contained in $\mathscr{M}$.

LEMMA 1.1. *If* $\mathscr{B} \not\subset \mathscr{M}$, *then there are units* f, g *on* $\mathscr{B}$ *and a basic sequence* $\mathscr{C}$ *which contains* $\mathscr{B}$ *such that* $f \circ_{\mathscr{C}} g$ *is not a unit on* $\mathscr{B}$.

For if $\mathscr{B} \not\subset \mathscr{M}$, then $\mathscr{B}$ contains some type II primitive pair, say (p, p). Take $\mathscr{C} = \mathscr{B}$ and let $f(n) = g(n) = 1(n = 1, 2, \cdots)$. Then f and g are units on $\mathscr{B}$, but $(f \circ_{\mathscr{C}} g)(p^t) = t + 1$, so $I(f \circ_{\mathscr{C}} g, \mathscr{B})$ does not exist.

A basic sequence $\mathscr{B}$ is said to be *cohesive* provided that for each positive integer k there is an integer $a > 1$ such that $(a, k) \in \mathscr{B}$. The characterization required above is given by

THEOREM 1.2. *Suppose $\mathscr{B} \subset \mathscr{M}$. In order that $f \circ_{\mathscr{E}} g$ be a unit on $\mathscr{B}$ for every pair f, g of units on $\mathscr{B}$ and every basic sequence $\mathscr{E}$ containing $\mathscr{B}$, it is both necessary and sufficient that $\mathscr{B}$ be cohesive.*

The following example will illustrate Theorem 1.2 and some of the preceding definitions.

EXAMPLE 1.3. Let $\{p_j\}_{j=1}^{\infty}$ be an enumeration of the primes in ascending order, and let $\mathscr{B}$ be the basic sequence generated by the following set of primitive pairs:

$$\bigcup_{j=1}^{\infty} \left\{ \bigcup_{i=1}^{\infty} (p_j, p_{j+1+i}) \right\} .$$

Thus $\mathscr{B}$ is generated by the pairs (2.5), (2, 7), (2, 11), $\cdots$, (3, 7), (3, 11), (3,13), $\cdots$, and so forth. $\mathscr{B} \subset \mathscr{M}$ and $\mathscr{B}$ is cohesive. Now define the arithmetic functions f and g by

$$f(k) = \begin{cases} 1 & \text{if } 6 \nmid k , \\ 2 & \text{if } 6 \mid k , \end{cases}$$

$$g(k) = \begin{cases} k & \text{if } 6 \nmid k , \\ m & \text{if } k = 6^a m, \text{ where } 6 \nmid m . \end{cases}$$

f and g are units on $\mathscr{B}$, and $f \circ_{\mathscr{E}} g$ is also a unit on $\mathscr{B}$ for every basic sequence $\mathscr{E}$ which contains $\mathscr{B}$.

We note that although neither f nor g is multiplicative with respect to $\mathscr{M}$, each is multiplicative with respect to $\mathscr{B}$. We will see shortly (Theorem 2.3) that the multiplicativity of f and g on $\mathscr{B}$ was unavoidable.

Theorem 1.2 provides a link between the multiplicative properties of certain arithmetic functions and the combinatorial properties (e.g., cohesiveness) of the underlying basic sequence. We discuss the function-theoretic consequences of cohesiveness in § 2 and use the results obtained there to give a proof of Theorem 1.2 in § 3.

In § 4 we examine the combinatorical aspects of cohesiveness and in § 5 we present two examples which supplement the results given in § 4.

2. Function-theoretic properties of cohesiveness. From the statement of Theorem 1.2 one is led naturally to look for other characterizations of cohesiveness in terms of the behavior of the units

on $\mathscr{B}$. A particularly simple criterion is given by

THEOREM 2.1. *A basic sequence $\mathscr{B}$ is cohesive if and only if, for every unit f on $\mathscr{B}$, $f(k) > 0$ for all k.*

Proof. Assume that $\mathscr{B}$ is cohesive and let a unit f on $\mathscr{B}$ and a positive integer k be given. Since $(1, k) \in \mathscr{B}$ and since there exists an integer $a > 1$ such that $(a, k) = (a, 1 \cdot k) \in \mathscr{B}$, it follows from [1, Theorem 4.11(a)] that $f(k) > 0$.

Conversely, suppose that $\mathscr{B}$ is not cohesive. Then there is an integer k_0 such that, for every $a > 1$, $(a, k_0) \notin \mathscr{B}$. Note that $k_0 > 1$. Consider the function f defined by $f(n) = 1$ if $n \neq k_0$, $f(k_0) = -1$. It is sufficient to show that f is a unit on $\mathscr{B}$.

Suppose $(u, v) \in \mathscr{B}$, $uv > k_0$. If either $u = k_0$ or $v = k_0$, say $v = k_0$, then $(u, v) = (u, k_0) \in \mathscr{B}$. That implies $u = 1$, so $uv = k_0$. But this contradicts the assumption $uv > k_0$. So $u \neq k_0$ and $v \neq k_0$, hence $f(u) = f(v) = 1$ and $\alpha_f(u, v) = 0$. It follows that $I(f, \mathscr{B}) = 0$ and f is a unit on $\mathscr{B}$.

Note that the exact value of $f(k_0)$ never comes into play in the second part of the proof of the preceding theorem. In fact, we have actually proved

THEOREM 2.2. *If $\mathscr{B}$ is not cohesive there exists an integer $k_0 > 1$ (which depends only on $\mathscr{B}$) such that, for any given value V, there is a unit f on $\mathscr{B}$ satisfying $f(k_0) = V$.*

The following necessary condition for cohesiveness is an immediate consequence of [1, Theorem 4.11].

THEOREM 2.3. *If $\mathscr{B}$ is a cohesive basic sequence, then every unit on $\mathscr{B}$ is multiplicative with respect to $\mathscr{B}$.*

COROLLARY 2.3.1. *If, for every unit f on $\mathscr{B}$, $f(k) > 0$ for all k, then every unit on $\mathscr{B}$ is multiplicative with respect to $\mathscr{B}$.*

The converse of Theorem 2.3 is not true; simply take $\mathscr{B} = \mathscr{S}$.

Theorem 2.1 illustrates the following line of reasoning: Suppose f has some multiplicative properties with respect to a basic sequence $\mathscr{B}$ and that $f(k)$ satisfies certain conditions for large values of k. Then if $\mathscr{B}$ is large enough, one might expect f to satisfy these conditions for all values of k. Another example is given by

THEOREM 2.4. *For any basic sequence $\mathscr{B}$, the following two statements are equivalent:*

  (1) *$\mathscr{B}$ is cohesive;*

(2) *every function f which satisfies*
  (a) $I(f, \mathcal{B}) = 0$,
  (b) $f(k) \neq 0$ for $k \geqq N$,
  also satisfies $f(k) \neq 0$ for all k.

Proof. Suppose $\mathcal{B}$ is cohesive and that f satisfies (a) and (b). Since $I(f, \mathcal{B}) = 0$ we may apply Lemma 2.3 of [1]. Let N_1 be the integer given there and let $N^* = \max\{N, N_1\}$. For any positive integer n, the cohesiveness of $\mathcal{B}$ implies the existence of an integer $a > 1$ such that $(a, n) \in \mathcal{B}$. If t is chosen so that $a^t \geqq N^*$, then $a^t n \geqq N$ and so $f(a^t n) \neq 0$. But also $a^t n \geqq N_1$, and since $(a^t, n) \in \mathcal{B}$, Lemma 2.3 of [1] implies $f(a^t)f(n) \neq 0$. Hence $f(n) \neq 0$ and (2) holds.

On the other hand, suppose $\mathcal{B}$ is not cohesive. Then by Theorem 2.2 (with $V = 0$), there is a unit f on $\mathcal{B}$ such that $f(k_0) = 0$. So (2) is false.

3. **Proof of Theorem 1.2.** Suppose first that $\mathcal{B}$ is cohesive. By Theorem 2.3, the units f and g are multiplicative with respect to $\mathcal{B}$. Hence, by Theorem 1 in [2], $f \circ_{\mathcal{E}} g$ is also multiplicative with respect to $\mathcal{B}$. Moreover, by Theorem 2.1, $f(k)$ and $g(k)$ are positive for all values of k, so $(f \circ_{\mathcal{E}} g)(k)$ is positive for every k. Therefore $f \circ_{\mathcal{E}} g$ is a unit on $\mathcal{B}$.

If $\mathcal{B}$ is not cohesive, then there is an integer $k_0 > 1$ such that $(a, k_0) \notin \mathcal{B}$ for each $a > 1$. Let p_0 be any prime which does not divide k_0 and define $\mathcal{E}$ to be the basic sequence generated by $\mathcal{B} \cup (p_0, k_0)$:

$$\mathcal{E} = \Gamma[\mathcal{B} \cup (p_0, k_0)] .$$

Since $(p_0, k_0) \notin \mathcal{B}$ and since p_0 and k_0 are relatively prime, we have $\mathcal{B} \subset \mathcal{E} \subset \mathcal{M}$.

We will show that there are units f, g on $\mathcal{B}$ for which $f \circ_{\mathcal{E}} g$ is not a unit on $\mathcal{B}$. In fact, let $g(n) = 1$ for all n and define f by

$$f(n) = \begin{cases} 1 \text{ if } n \neq k_0 , \\ 1 - 2^{\#}E_{k_0} \text{ if } n = k_0 , \end{cases}$$

where ${}^{\#}E_m$ is the number of pairs (a, b) in $\mathcal{E}$ for which $ab = m$.

Clearly g is a unit. The proof that f is a unit on $\mathcal{B}$ is essentially the second part of the proof of Theorem 2.1. To show that $f \circ_{\mathcal{E}} g$ is not a unit with respect to $\mathcal{B}$, we will show that

$$(f \circ_{\mathcal{E}} g)(p_0^t k_0) = \sum_{\substack{d \mid p_0^t k_0 \\ (d, p_0^t k_0 / d) \in \mathcal{E}}} f(d)g(p_0^t k_0 / d)$$

is zero for every $t \geqq 1$. We note first that if $d = p_0^s l$ where $l \mid k_0$, then either $s = 0$ or $s = t$ since $\mathscr{E} \subset \mathscr{M}$. Hence

$$(f \circ_{\mathscr{E}} g)(p_0^t k_0) = \sum_{\substack{l \mid k_0 \\ (l,\, k_0 p_0^t/l) \in \mathscr{E}}} f(l) + \sum_{\substack{l \mid k_0 \\ (p_0^t l,\, k_0/l) \in \mathscr{E}}} f(p_0^t l) .$$

Now $(p_0, k_0) \in \mathscr{E}$, so $(l, p_0^t) \in \mathscr{E}$ for every l which divides k_0. Hence

$$\left(l, \frac{k_0}{l} p_0^t\right) \in \mathscr{E} \Longleftrightarrow \left(l, \frac{k_0}{l}\right) \in \mathscr{E} .$$

Also, $(p_0^t, k_0/l) \in \mathscr{E}$ for every l which divides k_0, so

$$\left(p_0^t l, \frac{k_0}{l}\right) \in \mathscr{E} \Longleftrightarrow \left(l, \frac{k_0}{l}\right) \in \mathscr{E} .$$

It follows that

$$\begin{aligned}
(f \circ_{\mathscr{E}} g)(p_0^t k_0) &= \sum_{\substack{l \mid k_0 \\ (l, k_0/l) \in \mathscr{E}}} f(l) + \sum_{\substack{l \mid k_0 \\ (l, k_0/l) \in \mathscr{E}}} f(p_0^t l) \\
&= f(k_0) + \sum_{\substack{l \mid k_0,\, l < k_0 \\ (l, k_0/l) \in \mathscr{E}}} 1 + \sum_{\substack{l \mid k_0 \\ (l, k_0/l) \in \mathscr{E}}} 1 \\
&= f(k_0) + ({}^{\#}E_{k_0} - 1) + {}^{\#}E_{k_0} = 0 .
\end{aligned}$$

Thus $f \circ_{\mathscr{E}} g$ is not a unit on $\mathscr{B}$, and the proof of Theorem 1.2 is complete.

    An arithmetic function f is said to have *nonexceptional index* with respect to a basic sequence $\mathscr{B}$ if $I(f, \mathscr{B})$ exists and $0 < |I(f, \mathscr{B})| < 1$. Using the fact that f has nonexceptional index if and only if $f = F f^*$, where f^* is a unit on $\mathscr{B}$ and F is a constant satisfying $F > 0$, $F \neq 1$ (see Theorem 5.1 in [1]), we have the following straightforward consequence of Theorem 1.2.

    THEOREM 3.1. *Let $\mathscr{B}$ and $\mathscr{E}$ be basic sequences such that $\mathscr{B}$ is cohesive and $\mathscr{B} \subset \mathscr{E} \cap \mathscr{M}$. If f, and also g, is either a unit on $\mathscr{B}$ or has nonexceptional index, then*

    (1) $f \circ_{\mathscr{E}} g = H h^*$ *for some unit h^* on $\mathscr{B}$ and some constant $H > 0$,*

    (2) $f \circ_{\mathscr{E}} g$ *is either a unit or has nonexceptional index with respect to $\mathscr{B}$,*

    (3) $I(f \circ_{\mathscr{E}} g, \mathscr{B}) = \dfrac{I(f, \mathscr{B}) + I(g, \mathscr{B})}{1 + I(f, \mathscr{B}) I(g, \mathscr{B})}$.

    4. Combinatorial properties of cohesiveness. We will use the following notation:

$$P = \text{set of all primes,}$$

$$C_{\mathscr{B}}(m) = \{p \,|\, p \in P, \, (p, m) \in \mathscr{B}\} \,,$$
$$C_{\mathscr{B}}(A) = \bigcup_{a \in A} C_{\mathscr{B}}(a) \,.$$

A basic sequence which is cohesive is one which, roughly speaking, has enough primitive pairs and for which the primitive pairs are properly distributed. The following two theorems and their corollaries will make these notions more precise.

THEOREM 4.1. *If $\mathscr{B}$ is cohesive, then*

$$C_{\mathscr{B}}(C_{\mathscr{B}}(p)) = P$$

for every prime p.

Proof. Suppose $s \in P$. Since $\mathscr{B}$ is cohesive, there is a prime q_0 such that $(q_0, sp) \in \mathscr{B}$. But this means $(s, q_0) \in \mathscr{B}$ and $(q_0, p) \in \mathscr{B}$. Hence

$$s \in C_{\mathscr{B}}(q_0) \subset \bigcup_{q \in C_{\mathscr{B}}(p)} C_{\mathscr{B}}(q) = C_{\mathscr{B}}(C_{\mathscr{B}}(p)) \,.$$

We recall that a prime p is said to have *property F* with respect to the basic sequence $\mathscr{B}$ if p is finitely distributed with respect to $\mathscr{B}$ (i.e., $C_{\mathscr{B}}(p)$ is finite) and if every prime q in $C_{\mathscr{B}}(p)$ is also finitely distributed with respect to $\mathscr{B}$.

COROLLARY 4.1.1. *If $\mathscr{B}$ is cohesive, then no prime has property F with respect to $\mathscr{B}$.*

COROLLARY 4.1.2. *If $\mathscr{B}$ is cohesive, then $\mathscr{B}$ must contain infinitely many type I primitive pairs.*

COROLLARY 4.1.3. *Suppose $\mathscr{B}$ is cohesive. If there is a prime p_0 for which there is only one prime q such that $(q, p_0) \in \mathscr{B}$, then $(q, r) \in \mathscr{B}$ for every prime r.*

THEOREM 4.2. *If $\mathscr{B}$ is a cohesive basic sequence, then either $C_{\mathscr{B}}(p)$ is infinite for every prime p, or there is a prime q for which $C_{\mathscr{B}}(q) = P$.*

COROLLARY 4.2.1. *If $\mathscr{B}$ is cohesive and $\mathscr{B} \subset \mathscr{M}$, then $C_{\mathscr{B}}(p)$ is infinite for every prime p.*

Proof. We first note the obvious facts that (i) for any basic sequence $\mathscr{B}$, if $m \,|\, n$ then $C_{\mathscr{B}}(m) \supset C_{\mathscr{B}}(n)$ and (ii) if $\mathscr{B}$ is cohesive, then $C_{\mathscr{B}}(m)$ is not empty for any m.

Suppose $C_{\mathscr{B}}(p_0)$ is finite for some prime p_0. Enumerate the sequence of all primes: $p_1, p_2, \cdots$. Then

$$C_{\mathscr{B}}(p_0) \supset C_{\mathscr{B}}(p_0 p_1) \supset C_{\mathscr{B}}(p_0 p_1 p_2) \supset \cdots .$$

We thus have a nested, nonincreasing sequence of nonempty finite sets of primes, hence

$$A = \bigcap_{K=1}^{\infty} C_{\mathscr{B}}\left(\prod_{j=0}^{K} p_j\right)$$

is not empty. Every prime q in A satisfies $(q, p_j) \in \mathscr{B}, j = 1, 2, \cdots$. That proves the theorem.

The corollary follows from the fact that if $C_{\mathscr{B}}(q) = P$ for some prime q, then $(q, q) \in \mathscr{B}$, so $\mathscr{B}$ cannot be contained in $\mathscr{M}$.

5. Some examples. The theorems in the preceding section give us information (in terms of the primitive pairs in $\mathscr{B}$) about the size and structure of a cohesive basic sequence. The following two examples suggest that the structure of the primitive pairs in $\mathscr{B}$ is more important than the number of primitive pairs in determining whether $\mathscr{B}$ is cohesive.

We recall that the *density* of a basic sequence $\mathscr{B}$ is defined to be

$$\delta(\mathscr{B}) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{{}^{\#}B_k}{d(k)} ,$$

where ${}^{\#}B_k$ is the number of pairs (a, b) in $\mathscr{B}$ for which $ab = k$, and $d(k)$ is the number of divisors of k.

EXAMPLE 5.1. The basic sequence $\mathscr{B}$ generated by $\{(2, p) \mid p \in P\}$ is cohesive and $\delta(\mathscr{B}) = 0$. Thus a cohesive basic sequence may be very small if it contains the "right" primitive pairs.

EXAMPLE 5.2. Let P_1 and P_2 be two disjoint infinite sets of primes such that $P_1 \cup P_2 = P$, and let $\mathscr{B}$ be generated by $\{(p, q) \mid p, q \in P_1$ or $p, q \in P_2\}$. Clearly $\mathscr{B}$ is not cohesive. Moreover, by choosing the primes in P_2 to be large and very sparse, we can make $\delta(\mathscr{B})$ as close to 1 as we wish. So $\mathscr{B}$ can be very large, but without a suitable structure on the set of primitive pairs cohesiveness cannot be guaranteed.

We note that no prime is finitely distributed with respect to $\mathscr{B}$ and that $C_{\mathscr{B}}(p)$ is infinite for every prime p, hence the converses of Corollary 4.1.1 and Theorem 4.2 do not hold.

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