SCHLICHT MAPPINGS AND INFINITELY DIVISIBLE KERNELS

Roger Alan Horn
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The purpose of this note is to give a simple condition which is sufficient for a function on a real interval to be the boundary value of a schlicht (univalent) analytic mapping of the upper half plane into itself. This condition leads to a simple transformation which takes (possibly) non-schlicht mappings into schlicht ones. The methods used have applications to probability theory as well; they yield an interesting class of infinitely divisible characteristic functions.

We shall require some facts about infinitely divisible kernels; for a detailed exposition see [5]. If \( I \) is a real interval we denote by \( L_0(I) \) the set of all continuous complex valued functions which have compact support in \( I \) and whose integral over \( I \) vanishes. A continuous kernel \( K(x, y) \) on \( I \times I \) is said to be \textit{conditionally positive definite on} \( I \) if

\[
\iint_{I \times I} K(x, y)\phi(x)\bar{\phi}(y)\,dx\,dy \geq 0
\]

for all functions \( \phi \in L_0(I) \); it is said to be \textit{positive definite on} \( I \) if (1) is satisfied for all continuous functions \( \phi \) with compact support in \( I \); it is said to be \textit{infinitely divisible on} \( I \) if (for some fixed continuous determination of the argument) the kernel \( K^\alpha(x, y) \) is positive definite for all \( \alpha > 0 \).

The connection among these concepts is that a continuous Hermitian kernel \( K(x, y) \) with no zeroes is infinitely divisible on \( I \) if and only if (for some continuous determination of the argument) the kernel \( \log K(x, y) \) is conditionally positive definite on \( I \). If \( K(x, y) > 0 \) for all \( x, y \in I \) there is, of course, no difficulty about determining the argument. Finally, the relevance of these notions to function theory is indicated by the following result [6], [4]. If \( f \) is a differentiable function we define \( K_f(x, y) = [f(x) - f(y)]/(x - y) \) and agree that \( K_f(x, x) = f'(x) \).

**Theorem 1.** Let \( f \) be a continuously differentiable real valued function with positive derivative on a real interval \( I \). The function \( f \) possesses an analytic continuation onto the upper half plane which maps the upper half plane into itself if and only if the kernel \( K_f(x, y) \) is positive definite on \( I \). This mapping is schlicht if and only if \( K_f(x, y) \) is infinitely divisible on \( I \).
Although this result completely characterizes the boundary values of schlicht mappings, it is in practice much harder to verify that the kernel $K_f(x, y)$ is infinitely divisible than to test it for positive definiteness. By our remarks above, one must check whether $\log K_f(x, y)$ is conditionally positive definite, but the non-linearity of this expression in $f$ often leads to computational difficulties. In the following, we shall derive a more linear, and hopefully more useful, sufficient condition. Recall that a $C^\infty$ function $\phi$ defined on $(0, \infty)$ is completely monotonic if $(-1)^n\phi^{(n)}(x) \geq 0$ for all $x > 0$ and all $n = 1, 2, 3, \ldots$.

**Lemma 2.** Let $H(x, y)$ be a continuous Hermitian kernel on a real interval $I$ such that $\text{Re} \{H(x, y)\} > 0$ and such that $-H(x, y)$ is conditionally positive definite. If $\phi$ is any completely monotonic function then the kernel $\phi(H(x, y))$ is positive definite on $I$.

**Proof.** It is well known that a function $\phi$ is completely monotonic if and only if there exists a nonnegative measure $d\mu$ such that $\phi(x) = \int_0^\infty e^{-sx}d\mu(s)$ for all $x > 0$ ([8], p. 160); in this event $\phi$ is analytic in the whole right half plane. But since $\text{Re} \{H(x, y)\} > 0$ and $\exp(-sH(x, y))$ is positive definite (even infinitely divisible) for all $s > 0$, it follows that $\phi(H(x, y)) = \int_0^\infty \exp(-sH(x, y))d\mu(s)$ is convergent and is a positive definite kernel.

An infinitely divisible completely monotonic function $\phi$ is a function such that $\phi^\alpha$ is completely monotonic for all $\alpha > 0$; if $\phi \equiv 0$, a necessary and sufficient condition for this is that the derivative of $-\ln \phi$ be completely monotonic ([3], p. 229). Using the lemma and the definition of an infinitely divisible kernel we obtain

**Corollary 3.** Let $\phi$ be a positive differentiable function on $(0, \infty)$ such that $-\phi'/\phi$ is completely monotonic, and suppose $H(x, y)$ satisfies the hypotheses of Lemma 2. Then $\phi(H(x, y))$ is an infinitely divisible kernel.

Since the function $\phi(x) = 1/x$ satisfies this condition, the following result is immediate.

**Corollary 4.** If $H(x, y)$ satisfies the conditions of Lemma 2, then the kernel $1/H(x, y)$ is infinitely divisible.

Now suppose that $g$ is a continuously differentiable real valued function on a real interval $I$, so that $K_g(x, y)$ is a continuous symmetric kernel. If $g'(x) > 0$ on $I$ then $K_g(x, y)$ is a positive kernel and the
inverse function $g^{-1}$ is defined on the interval $g(I)$. Thus, if we assume that $-K_x(x, y)$ is conditionally positive definite on $I$ then we conclude from Corollary 4 that the kernel

$$\frac{1}{K_x(x, y)} = \frac{x - y}{g(x) - g(y)} = \frac{g^{-1}(g(x)) - g^{-1}(g(y))}{g(x) - g(y)}$$

is infinitely divisible. But this is equivalent to the kernel $K_{g^{-1}}(s, t)$ being infinitely divisible on $g(I)$ and so we may apply Theorem 1 to obtain the conclusion of the following

**Theorem 5.** Let $g$ be a continuously differentiable real valued function with positive derivative on a real interval $I$ and suppose that the kernel

$$-K_x(x, y) = -\frac{g(x) - g(y)}{x - y}$$

is conditionally positive definite on $I$. Then the inverse function $g^{-1}$ has an analytic continuation from $g(I)$ onto the upper half plane which is a schlicht mapping of the upper half plane into itself.

Thus, to ensure that a real function $f$ on a real interval $I$ is the boundary value of a schlicht self-mapping of the upper half plane it is sufficient to check that $f'(x) > 0$ and that $-K_{f^{-1}}(x, y)$ is conditionally positive definite on $f^{-1}(I)$.

The crucial condition in Theorem 5 is that the kernel $-K_x(x, y)$ be conditionally positive definite, and a great deal is known about functions which satisfy this condition. For example, they are real analytic and are analytically continuable onto the upper half plane, they have a simple integral representation, and they arise as the infinitesimal transformations of the pseudo-semigroup $\mathcal{M}_\omega$ of self-mappings of the upper half plane which have real boundary values on $I$ ([6] and [2], pp. 53-54). Furthermore, it is easy to find many non-trivial functions which satisfy this condition. Denote by $\mathcal{M}_\omega(0)$ the class of functions $f$ which are analytic in the upper half plane, map it into itself, are real valued on some open real interval containing zero, and are normalized by the condition $f(0) = 0$.

**Lemma 6.** Let $a$ be a real number, let $b \geq 0$ and let $f \in \mathcal{M}_\omega(0)$. Then the functions $g_0(x) = a$, $g_1(x) = ax$, $g_2(x) = ax^2$, and $g_3(x) = bx^2f(x)$ are such that $K_{g_i}(x, y)$ is conditionally positive definite on some neighborhood of the origin, $i = 0, 1, 2, 3$.

**Proof.** This follows from a direct computation for $i = 0, 1, 2$
but for $i = 3$ we need to know ([1], p. 63) that $f \in \mathcal{M}_\infty(0)$ if and only if

\begin{equation}
    f(x) = \int_{-1}^1 \frac{x}{1 - tx} d\mu(t)
\end{equation}

for some $\epsilon > 0$ and some nonnegative bounded measure $d\mu$ on $[-\epsilon, \epsilon]$. Thus, since the assertion for $i = 3$ follows for the special case $f(x) = x/(1 - tx)$ by direct computation, it follows for all $f \in \mathcal{M}_\infty(0)$ by linearity.

Using the four types of functions introduced in this lemma we can now use Theorem 5 to construct a wide class of schlicht mappings.

**Theorem 7.** Let $f \in \mathcal{M}_\infty(0)$, let $a_i > 0$, $a_3 \geq 0$, and let $a_0$ and $a_2$ be real numbers. Then the function

\[ g(x) = a_0 + a_1 x + a_2 x^2 - a_3 x^3 f(x) \]

is such that the inverse function $g^{-1}$ has an analytic continuation from a real neighborhood of $a_0$ onto the upper half plane which is a schlicht mapping of the upper half plane into itself.

**Proof.** The kernel $-K_s(x, y)$ is conditionally positive definite by Lemma 6 and $g'(x) > 0$ in some real neighborhood of zero. The result follows from Theorem 5.

Although this construction provides a wealth of schlicht mappings, it is far from exhaustive: the functions $f(z) = 3[\sqrt{z} + 1 - 1]$ and $f(z) = \log (z + 1)$ are schlicht mappings which are not of this form.

**Remark 1.** Linear combinations of the four functions in Lemma 6 are in fact the only smooth functions $g$ such that $K_s(x, y)$ is conditionally positive definite. In order to prove this we use the following criterion for a kernel to be conditionally positive definite.

**Lemma 8.** Let $H(x, y)$ be a continuous kernel on a real interval $I$ and let $x_0 \in I$. Then $H(x, y)$ is conditionally positive definite on $I$ if and only if the kernel

\[ H^*_s(x, y) = H(x, y) - H(x, x_0) - H(x_0, y) + H(x_0, x_0) \]

is positive definite on $I$.

**Proof.** If $\phi \in L_0(I)$, then

\[ \int_I \int_I H^*_s(x, y)\phi(x)\overline{\phi(y)}dxdy = \int_I \int_I H(x, y)\phi(x)\overline{\phi(y)}dxdy . \]
and hence $H(x, y)$ is conditionally positive definite if $H^*_x(x, y)$ is positive definite. Conversely, suppose $H(x, y)$ is conditionally positive definite and let $\{f_n(x)\}$, $n = 1, 2, 3, \ldots$ be an approximate identity based at $x_0$, i.e., each $f_n$ is a nonnegative continuous function with support in $I \cap [x_0 - n^{-1}, x_0 + n^{-1}]$ and $\int_I f_n(x) dx = 1$ for all $n$. If $\phi$ is any continuous function with compact support in $I$, let $\phi_n(x) = \phi(x) - f_n(x) \int_I \phi(t) dt$ and observe that $\phi_n \in L_0(I)$ for all large $n$. Thus,

$$0 \leq \iint_{I \times I} H(x, y) \phi_n(x) \bar{\phi}_n(y) dx dy$$

$$= \iint_{I \times I} \left\{ H(x, y) - \int_I H(x, t)f_n(t) dt - \int_I H(s, y)f_n(s) ds \right\} \phi(x) \bar{\phi}(y) dx dy$$

$$\rightarrow \iint_{I \times I} \left\{ H(x, y) - H(x, x_0) - H(x_0, y) + H(x_0, x_0) \phi(x) \bar{\phi}(y) \right\} dx dy$$

$$= \iint_{I \times I} H^*_x(x, y) \phi(x) \bar{\phi}(y) dx dy$$

as $n \to \infty$. Since $\phi$ is arbitrary, we conclude that the kernel $H^*_x(x, y)$ must be positive definite.

**Lemma 9.** Let $K(x, y)$ be a continuous kernel on a real interval $I$. Then $xyK(x, y)$ is positive definite kernel if and only if $K(x, y)$ is a positive definite kernel.

**Proof.** If zero is not a point of $I$ this is trivial, so suppose $0 \in I$, let $\varepsilon > 0$, and denote by $f_\varepsilon$ the unique even function such that

$$f_\varepsilon(x) \equiv \begin{cases} 
0 & \text{if } x \in [0, \varepsilon] \\
\varepsilon^{-1}(x - \varepsilon) & \text{if } x \in [\varepsilon, 2\varepsilon] \\
1 & \text{if } x \geq \varepsilon.
\end{cases}$$

Let $M = \sup_{x,y} |K(x, y)|$, let $\phi$ be a continuous function with compact support in $I$, and assume that $xyK(x, y)$ is positive definite on $I$. Then

$$\iint_{I \times I} K(x, y) \phi(x) \bar{\phi}(y) dx dy$$

$$= \iint_{I \times I} K(x, y) \phi(x) \bar{\phi}(y)(1 - f_\varepsilon(y)) dx dy$$

$$+ \iint_{I \times I} K(x, y) \phi(x) \bar{\phi}(y)f_\varepsilon(y)(1 - f_\varepsilon(x)) dx dy$$

$$+ \iint_{I \times I} K(x, y) \phi(x) f_\varepsilon(x) \bar{\phi}(y)f_\varepsilon(y) dx dy$$
\[
\geq - \int_{I \times I} |K(x, y)\phi(x)\bar{\phi}(y)(1 - f_\epsilon(y))| \, dx \, dy \\
- \int_{I \times I} |K(x, y)\bar{\phi}(y)f_\epsilon(y)\phi(x)(1 - f_\epsilon(x))| \, dx \, dy \\
+ \int_{I \times I} xyK(x, y)x^{-1}\phi(x)f_\epsilon(x)y^{-1}\bar{\phi}(y)f_\epsilon(y) \, dx \, dy
\]
\[
\geq - 6M\epsilon \sup_{I} |\phi(x)| \int_{I} |\phi(x)| \, dx.
\]

For the last inequality we have used the hypothesis that \(xyK(x, y)\) is positive definite and the fact that the function \(x^{-1}\phi(x)f_\epsilon(x)\) is a continuous function with compact support in \(I\). Since \(\epsilon > 0\) is arbitrary we conclude that
\[
\int_{I \times I} K(x, y)\phi(x)\bar{\phi}(y) \, dx \, dy \geq 0,
\]
i.e., \(K(x, y)\) is positive definite. The converse is trivial.

Now assume that the kernel \(H(x, y)\) is of the special form \(H(x, y) = K_g(x, y)\), where \(g\) is a real valued function which is three times continuously differentiable on an open real interval containing zero. Assume that \(g(0) = g'(0) = g''(0) = 0\). Then \(g(x)/x^2\) is continuously differentiable and
\[
H_\epsilon(x, y) = K_g(x, y) - K_g(x, 0) - K_g(0, y) + K_g(0, 0)
\]
\[
= xy - \frac{x^2}{y^2} = xyK_h(x, y),
\]
where we set \(h(x) \equiv g(x)/x^2\). Thus, Lemma 8 says that \(K_g(x, y)\) is conditionally positive definite if and only if \(xyK_h(x, y)\) is positive definite, and Lemma 9 says this is equivalent to the kernel \(K_h(x, y)\) being positive definite. We conclude that \(K_g(x, y)\) is conditionally positive definite if and only if \(K_h(x, y)\) is positive definite. But this means that \(h(x) = g(x)/x^2 \in \mathcal{M}_\infty(0)\) and hence \(h\) has the integral representation (2). The normalization we assumed for \(g\) can always be attained by subtracting a suitable quadratic polynomial, since Lemma 6 shows that every such polynomial has a conditionally positive definite difference quotient kernel. We summarize our results as

**Theorem 10.** Let \(g\) be real valued function on an open real interval \(I\) containing zero. The following are equivalent:

(a) The function \(g\) is three times continuously differentiable and the kernel \(K_g(x, y) = [g(x) - g(y)]/(x - y)\) is conditionally positive
The function $g$ has the form

$$g(x) = \alpha_0 + a_1x + a_2x^2 + a_3x^2f(x)$$

where $\alpha_0, a_1, a_2$ are real numbers, $a_3 \geq 0$ and $f \in \mathcal{F}_0$. 

The function $g$ has the form

$$g(x) = a_0 + a_1 + a_2x^2 + \int_{-\infty}^{\infty} \frac{x^3d\mu}{1-\varepsilon x^2}$$

where $a_0, a_1, a_2$ are real numbers, $\varepsilon \geq 0$ and $d\mu$ is a nonnegative bounded measure.

It should be noted that it is sufficient in (a) to assume only that $g$ is continuously differentiable; the condition on the kernel then implies that $g$ is analytic [6]. This characterization of the functions $g$ such that $K_g(x, y)$ is conditionally positive definite was obtained first by C. FitzGerald [2] using less elementary results on analytic kernels.

REMARK 2. Lemma 2 and its corollaries are also useful in probability theory where one is interested in continuous Hermitian kernels of the form $K(x, y) = f(x - y)$, $f(0) = 1$. Such a kernel is positive definite if and only if $f(x)$ is the Fourier transform of a (unique) probability measure on the line, i.e., $f(x)$ is a characteristic function; this kernel is infinitely divisible if and only if the measure is infinitely divisible. If $f(x)$ is the characteristic function of an infinitely divisible probability measure, then the kernel $H(x, y) = lnf(x - y)$ is conditionally positive definite and has nonpositive real part since $|f(x)| \leq 1$ for all real $x$. Thus, the kernel $H(x, y) = \lambda - lnf(x - y)$ satisfies the hypotheses of Lemma 2 if $\lambda > 0$ and hence the kernel

$$\frac{\lambda}{\lambda - lnf(x - y)}$$

is infinitely divisible by Corollary 4. But this means that the function $\phi(x) = \lambda/(\lambda - lnf(x))$ is an infinitely divisible characteristic function whenever $f$ is an infinitely divisible characteristic function and $\lambda > 0$. This result was obtained by F. W. Steutel [7] from very different considerations.

REFERENCES


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