SUBALGEBRA SYSTEMS OF POWERS OF PARTIAL UNIVERSAL ALGEBRAS

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A set \( A \) and an integer \( n > 1 \) are given. \( S \) is any family of subsets of \( A^n \). Necessary and sufficient conditions are found for the existence of a set \( F \) of finitary partial operations on \( A \) such that \( S \) is the set of all subalgebras of \( \langle A; F \rangle^n \). As a corollary, a family \( E \) of equivalence relations on \( A \) is the set of all congruences on \( \langle A; F \rangle \) for some \( F \) if and only if \( E \) is an algebraic closure system on \( A^2 \).

For any partial universal algebra, the subalgebras of its \( n \)th direct power form an algebraic lattice. The characterization of such lattices for the case \( n = 1 \) was essentially given by G. Birkhoff and O. Frink [1]. For the case \( n = 2 \), the characterization was given by the author [4] (see also [3]). The connection between the subalgebra lattices of partial universal algebras and their direct squares was described by the author [5].

In the present paper we are concerned with the subalgebra systems from the following point of view: given a set \( A \) and a positive integer \( n \), which systems of subsets of \( A^n \) are the subalgebra systems of \( \langle A; F \rangle^n \) for some set of partial operations \( F \) on \( A \)? The problem where \( F \) is required to be full is Problem 19 of G. Gratzer [2]. For \( n = 1 \), such systems are precisely the algebraic closure systems on \( A[1] \). The description of the case \( n \geq 2 \) is given here by the Characterization Theorem. We also show that there are partial universal algebras \( \langle A; F \rangle \) such that the subalgebra system of \( \langle A; F \rangle^2 \) is not equal to the subalgebra system of \( \langle A; G \rangle^2 \) for any set of full operations \( G \). The methods of this paper can be modified to get similar results for infinitary partial algebras, the arities of whose operations are less than a given infinite ordinal.

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a nonvoid subset of \( A \) which is closed under all elements of \( F \). We denote the set of all subalgebras of a partial algebra \( \langle A; F \rangle \) by \( S(\langle A; F \rangle) \) and we will consider \( \phi \in S(\langle A; F \rangle) \) if and only if the intersection of all nonvoid subalgebras of \( \langle A; F \rangle \) is empty.

**Proposition 1.** \( S(\langle A; F \rangle^n) \) is an algebraic closure system on \( A^n \).

If \( n = 1 \), this follows from the result of G. Birkhoff and O. Frink [1]. For any positive \( n \), \( S(\langle A; F \rangle^n) = S(\langle A^n; F \rangle) \).

We shall consider only the case \( n \geq 2 \).

2. Let \( S_n \) be the group of all permutations of \( \{1, \cdots, n\} \). Denote by \( P(A^n) \) the set of all subsets of \( A^n \). If \( s \in S_n \) and \( B \in P(A^n) \), we define

\[
B_s = \{ a: a \in A^n, b, b \in B, a(i) = b(is^{-1}), 1 \leq i \leq n \}.
\]

For \( n = 2 \), \( B \subseteq A^2 \), \( B(12) \) is the inverse binary relation of \( B \).

**Proposition 2.** The mapping which associates to every \( s \in S_n \) the operator on \( P(A^n) \) defined by (1) is a group homomorphism of \( S_n \) into the group of all automorphisms of the lattice \( \langle S(\langle A; F \rangle^n); \subseteq \rangle \).

3. Let \( \alpha \) be a nonvoid subset of \( \{1, \cdots, n\} \), \( i = \min \alpha \) and \( B \in P(A^n) \). Define

\[
B\alpha = \{ a: a \in A^n, b \in B, a(j) = b(j) \text{ if } j \notin \alpha, a(j) = b(i) \text{ if } j \in \alpha, 1 \leq j \leq n \}.
\]

It is easy to verify that if \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) then

\[
B\{i_1, \cdots, i_k\} = (\cdots((B\{i_1, i_2\} \{i_2, i_3\}) \cdots) \{i_{k-1}, i_k\}.
\]

If \( C \in P(A^n) \), we denote by \( F(C) \) the subalgebra of \( \langle A; F \rangle^n \) generated by \( C \).

**Proposition 3.** If \( C \in P(A^n) \), \( \alpha - \) a nonvoid subset of \( \{1, \cdots, n\} \) and \( s \in S_n \), then

\[
F(C)\alpha \subseteq F(C\alpha)
\]

\[
F(C)s = F(Cs).
\]

4. We denote by \( A^k \) the diagonal of \( A^k \) \( B \times A^n \) will be identified with \( B \).

The Characterization Theorem. Let \( S \subseteq P(A^n) \). \( S = S(\langle A; F \rangle^n) \) for some set of finitary partial operations \( F \) if and only if
(a) $S$ is an algebraic closure system on $A^n$
(b) if $B \in S$, $1 \leq i < j \leq n$, then $B(ij) \in S$.
(c) $\Delta_n \times A^{n-2} \in S$
(d) $[C] \{1, 2\} \subseteq [C(1, 2)]$ for all nonvoid finite $C \in P(A^n)$
(e) if $\phi \in S$, then $\phi = \cap \{B : \phi \neq B \in S\}$.

([C] denotes the intersection of all elements of $S$ containing $C$).

It can be shown that conditions (a), (b), (c), (d) and (e) are independent.

It is clear that $\Delta_n \times A^{n-2}$ is a subalgebra of $\langle A; F^* \rangle$ for all $F$.
That conditions (a), (b) and (d) are necessary follows from Propositions 1, 2 and 3.

**Proof of Sufficiency.** For every positive integer $m$ and every ordered $m + 1$-tuple $(a_1, \ldots, a_m, a)$ of elements of $A^n$ such that $a \in [a_1, \ldots, a_m]$ we associate an $m$-ary partial operation $f$ on $A$ such that

$$Df = \text{domain of definition of } f = \{(a_i(i), \ldots, a_m(i)) : 1 \leq i \leq n\}$$

and

$$a_1(i) \cdots a_m(i) f = a(i), 1 \leq i \leq n.$$  

Let $F$ be the set of all such finitary partial operations.

The following lemmas constitute the proof of sufficiency:

**LEMMA 1.** If $C \in P(A^n)$, $s \in S_n$, then $[C]_s = [Cs]$.

By (a), $S$ is a closure system hence $[C] \in S$. From (b) $[C](ij) \in S$ for all $1 \leq i < j \leq n$. Hence, by Proposition 2, $[C]_s \in S (P(A^n) = S(\langle A; \phi^* \rangle))$. But

$$Cs \subseteq [C]_s \in S.$$

Hence

$$[Cs] \subseteq [C]_s.$$

Also

$$C = (Cs)s^{-1}.$$

Hence

$$[C] = [(Cs)s^{-1}] \subseteq [Cs]s^{-1}.$$

And so

$$[C]_s \subseteq [Cs].$$

**LEMMA 2.** If $\alpha$ is a nonvoid subset of $\{1, \ldots, n\}$ and $C \in P(A^n)$, $C$ is finite and nonvoid, then

$$[C]_\alpha \subseteq [C\alpha].$$
First we show Lemma 2 for the case $\alpha = \{i, j\}, 1 \leq i < j \leq n$.

$$
[C] \{i, j\} = ([C(1i)(2j)] \{1, 2\})(1i)(2j) \\
= ([C(1i)(2j)] \{1, 2\})(1i)(2j) \quad \text{(by Lemma 1)} \\
\subseteq [(C(1i)(2j)] \{1, 2\})(1i)(2j) \quad \text{by (d)} \\
= [(C(1i)(2j)] \{1, 2\})(1i)(2j)] \\
= [C\{i, j\}] .
$$

If $1 \leq i_1 < \cdots < i_k \leq n$, then

$$
[C] \{i_1, \cdots, i_k\} = \ldots (C[i_{k-1}, i_k]) \{i_{k-1}, i_k\} \quad \text{(by (3))} \\
\subseteq \ldots \cdot \{1, \cdots, 2\}(1i)(2j) \in S \quad \text{(by (c) and (b)) for all } 1 \leq i \leq m.
$$

**LEMMA 3.** *The definition of $F$ is correct, i.e. every $f \in F$ is one valued.*

Lemma 3 will be established once we show that whenever $a_1, \cdots, a_m \in A^n, f \in F$ are such that $a_1(i) \cdots a_m(i)f$ is defined for every $1 \leq i \leq n$ and if for some $1 \leq p < q \leq n$ $a_i(p) = a_i(q), \cdots, a_m(p) = a_m(q)$; then

$$
a_1(p) \cdots a_m(p)f = a_1(q) \cdots a_m(q)f .
$$

By the definition of $F$, there are $c_1, \cdots, c_m, c \in A^n$ such that

$$
c \in \{c_1, \cdots, c_m\},
$$

$$
Df = \{(c_1(i), \cdots, c_m(i)): 1 \leq i \leq n\}
$$

and

$$
c_1(i) \cdots c_m(i)f = c(i); 1 \leq i \leq n .
$$

Hence

$$
\{(a_1(i), \cdots, a_m(i)): 1 \leq i \leq n\} \subseteq Df
$$

$$
= \{(c_1(i), \cdots, c_m(i)): 1 \leq i \leq n\} .
$$

So there are $s \in S_n$ and $\alpha$ nonvoid subset of $\{1, \cdots, n\}$ such that

$$
\alpha t = c, s\alpha, 1 \leq t \leq m .
$$

Since every $a_t$ satisfies $a_t(p) = a_t(q)$. We have $a_t \in (A_2 \times A^{n-2})(1p)(2q) \in S$ (by (c) and (b)) for all $1 \leq t \leq m$. Then

$$
\{c_1, \cdots, c_m\} s\alpha = \{a_1, \cdots, a_m\} \subseteq (A_2 \times A^{n-2})(1p)(2q) .
$$

But

$$
[[c_1, \cdots, C_m]] s\alpha = [[c_1, \cdots, c_m]] s\alpha
$$
Define $a \in A^n$ by

$$a(j) = a_1(j) \cdots a_m(j) \cdot f \quad 1 \leq j \leq n.$$ 

Then

$$a = cs\alpha \in [(a_1, \cdots, a_m)]s\alpha \subseteq (\Delta \times A^{n-2})(1p)(2q).$$

Hence

$$a_1(p) \cdots a_m(p) \cdot f = a(p) = a(q) = a_1(q) \cdots a_m(q) \cdot f.$$ 

**Lemma 4.** If $\phi \neq B \in S(\langle A; F^* \rangle^n)$ then $B \in S.$

Since $S$ is an algebraic closure system it will be sufficient to show that if $C$ is a finite nonvoid subset of $B,$ the $[C] \subseteq B.$

Let $b_1, \cdots, b_m \in B$ and $b \in [b_1, \cdots, b_m]$. By the definition of $F,$ there is $f \in F'$ such that $b_1(i) \cdots b_m(i) \cdot f$ is defined and is equal to $b(i)$ for all $1 \leq i \leq n$. $B$ is a subalgebra of $\langle A; F^* \rangle^n,$ hence $b \in B.$

**Lemma 5.** If $\phi \neq B \in S$ then $B \in S(\langle A; F^* \rangle^n).$

Let $f \in F'$; $a_1, \cdots, a_m \in B$ and $a_1(i) \cdots a_m(i) \cdot f = a(i)$, $1 \leq i \leq n$. We must show that $a \in B$.

By the definition of $F'$ there are $c_1, \cdots, c_m, c \in A^n$ such that $c \in [[c_1, \cdots, c_m]]$

$$Df = \{(c_1(i), \cdots, c_m(i)): 1 \leq i \leq n\}$$

and

$$c_1(i) \cdots c_m(i) \cdot f = c(i), 1 \leq i \leq n.$$ 

So

$$\{(a_1(i), \cdots, a_m(i)): 1 \leq i \leq n\} \subseteq Df = \{(c_1(i), \cdots, c_m(i)): 1 \leq i \leq n\}.$$ 

As in Lemma 3

$$a_t = c_i s\alpha, 1 \leq t \leq m; a = csd,$$

for some $s \in S_n$ and $\phi \neq a \subseteq \{1, \cdots, n\}.$

But

$$c \in [[c_1, \cdots, c_m]].$$

Hence

$$a = cs\alpha \in [[c_1, \cdots, c_m]]s\alpha \subseteq [[c_1, \cdots, c_m]]s\alpha$$

$$= [[a_1, \cdots, a_m]] \subseteq B.$$ 

**Theorem 5.** Let $C \subseteq P(A^2)$. $C$ is the set of all congruence
relations on \( \langle A; F \rangle \) for some set of finitary partial operations \( F \) if and only if \( C \) is an algebraic closure system on \( A^2 \) and every element of \( C \) is an equivalence relation on \( A \).

That the set of all congruence relations on \( \langle A; F \rangle \) is an algebraic closure system on \( A^2 \) is well known.

If \( C \subseteq P(A^2) \) is a set of equivalence relations which is also an algebraic closure system on \( A^2 \) then \( C \) satisfies all the conditions (a), (b), (c), (d) and (e) of the Characterization Theorem. Hence \( C = S(\langle A; F' \rangle^2) \) for some set of finitary partial operations \( F \). Since every element of \( C \) is an equivalence relation on \( A \) and a subalgebra of \( \langle A; F' \rangle^2 \), it is a congruence relation on \( \langle A; F' \rangle \). Since a congruence relation on \( \langle A; F' \rangle \) is an equivalence relation on \( A \) which is also a subalgebra of \( \langle A; F' \rangle^2 \), the Theorem is proved.

6. The following proposition shows that our Characterization Theorem does not solve the corresponding problem for full algebras.

**Proposition 4.** There are partial algebras \( \langle A; F \rangle \) such that \( S(\langle A; F' \rangle^2) \neq S(\langle A; G' \rangle^2) \) for any set of full finitary operations \( G \).

Let \( A = \{1, 2, 3\} \), \( F = \{f_1, f_2, f_3, g\} \); \( f_1, f_2, f_3 \) are full unary operations, \( g \) is a partial binary operation.

\( f_i \) is the constant function taking the value \( i \), \( i = 1, 2, 3 \).

\[
Dg = \{(1, 2), (2, 1)\} \\
12g = 3, (21)g = 2, \\
B = A_1 \cup \{(1, 2)\}, C = A_2 \cup \{(2, 1)\} \\
BoC = B \cup C, \\
B, C \in S(\langle A; F' \rangle^2), \text{ but} \\
BoC \notin S(\langle A; F' \rangle^2)
\]

since any subalgebra of \( \langle A; F' \rangle^2 \) containing \( (1, 2) \) and \( (2, 1) \) contains also \( (3, 2) \) and \( (2, 3) \).

**References**


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