

# Pacific Journal of Mathematics

## **SUBALGEBRA SYSTEMS OF POWERS OF PARTIAL UNIVERSAL ALGEBRAS**

A. A. ISKANDER

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A set  $A$  and an integer  $n > 1$  are given.  $S$  is any family of subsets of  $A^n$ . Necessary and sufficient conditions are found for the existence of a set  $F$  of finitary partial operations on  $A$  such that  $S$  is the set of all subalgebras of  $\langle A; F \rangle^n$ . As a corollary, a family  $E$  of equivalence relations on  $A$  is the set of all congruences on  $\langle A; F \rangle$  for some  $F$  if and only if  $E$  is an algebraic closure system on  $A^2$ .

For any partial universal algebra, the subalgebras of its  $n$ th direct power form an algebraic lattice. The characterization of such lattices for the case  $n = 1$  was essentially given by G. Birkhoff and O. Frink [1]. For the case  $n = 2$ , the characterization was given by the author [4] (see also [3]). The connection between the subalgebra lattices of partial universal algebras and their direct squares was described by the author [5].

In the present paper we are concerned with the subalgebra systems from the following point of view: given a set  $A$  and a positive integer  $n$ , which systems of subsets of  $A^n$  are the subalgebra systems of  $\langle A; F \rangle^n$  for some set of partial operations  $F$  on  $A$ ? The problem where  $F$  is required to be full is Problem 19 of G. Gratzner [2]. For  $n = 1$ , such systems are precisely the algebraic closure systems on  $A$ [1]. The description of the case  $n \geq 2$  is given here by the Characterization Theorem. We also show that there are partial universal algebras  $\langle A; F \rangle$  such that the subalgebra system of  $\langle A; F \rangle^2$  is not equal to the subalgebra system of  $\langle A; G \rangle^2$  for any set of full operations  $G$ . The methods of this paper can be modified to get similar results for infinitary partial algebras, the arities of whose operations are less than a given infinite ordinal.

The author is grateful to the referee for a number of valuable indications.

1. Let  $A$  be a set and  $n$  be a positive integer. The set of all functions from  $\{1, \dots, n\}$  into  $A$  will be denoted by  $A^n$ . If  $F$  is a set of finitary partial operations on  $A$ , the partial algebra structure obtained on  $A$  will be denoted by  $\langle A; F \rangle$ . By  $\langle A; F \rangle^n$  we will mean the partial algebra  $\langle A^n; F \rangle$  such that if  $f \in F$  is an  $m$ -ary partial operation and  $a_1, \dots, a_m \in A^n$  then  $a_1 \dots a_m f$  is defined and equal to  $a \in A^n$  if and only if  $a_1(j) \dots a_m(j) f$  is defined and is equal to  $a(j)$  for all  $1 \leq j \leq n$ . By a subalgebra of a partial algebra  $\langle A; F \rangle$  we will mean

a nonvoid subset of  $A$  which is closed under all elements of  $F$ . We denote the set of all subalgebras of a partial algebra  $\langle A; F \rangle$  by  $S(\langle A; F \rangle)$  and we will consider  $\phi \in S(\langle A; F \rangle)$  if and only if the intersection of all nonvoid subalgebras of  $\langle A; F \rangle$  is empty.

PROPOSITION 1.  $S(\langle A; F \rangle^n)$  is an algebraic closure system on  $A^n$ .

If  $n = 1$ , this follows from the result of G. Birkhoff and O. Frink [1]. For any positive  $n$ ,  $S(\langle A; F \rangle^n) = S(\langle A^n; F \rangle)$ .

We shall consider only the case  $n \geq 2$ .

2. Let  $S_n$  be the group of all permutations of  $\{1, \dots, n\}$ . Denote by  $P(A^n)$  the set of all subsets of  $A^n$ . If  $s \in S_n$  and  $B \in P(A^n)$ , we define

$$(1) \quad Bs = \{a: a \in A^n, b, b \in B, a(i) = b(is^{-1}), 1 \leq i \leq n\}.$$

For  $n = 2$ ,  $B \subseteq A^2$ ,  $B(12)$  is the inverse binary relation of  $B$ .

PROPOSITION 2. The mapping which associates to every  $s \in S_n$  the operator on  $P(A^n)$  defined by (1) is a group homomorphism of  $S_n$  into the group of all automorphisms of the lattice  $\langle S(\langle A; F \rangle^n); \subseteq \rangle$ .

3. Let  $\alpha$  be a nonvoid subset of  $\{1, \dots, n\}$ ,  $i = \min \alpha$  and  $B \in P(A^n)$ . Define

$$(2) \quad B\alpha = \{a: a \in A^n, b \in B, a(j) = b(j) \text{ if } j \notin \alpha, \\ a(j) = b(i) \text{ if } j \in \alpha, 1 \leq j \leq n\}.$$

It is easy to verify that if  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  then

$$(3) \quad B\{i_1, \dots, i_k\} = (\dots((B\{i_1, i_2\})\{i_2, i_3\})\dots)\{i_{k-1}, i_k\}.$$

If  $C \in P(A^n)$ , we denote by  $F(C)$  the subalgebra of  $\langle A; F \rangle^n$  generated by  $C$ .

PROPOSITION 3. If  $C \in P(A^n)$ ,  $\alpha$  — a nonvoid subset of  $\{1, \dots, n\}$  and  $s \in S_n$ , then

$$(4) \quad F(C)\alpha \subseteq F(C\alpha)$$

$$(5) \quad F(C)s = F(Cs) .$$

4. We denote by  $\Delta_k$  the diagonal of  $A^k, B \times A^\circ$  will be identified with  $B$ .

*The Characterization Theorem.* Let  $S \subseteq P(A^n)$ .  $S = S(\langle A; F \rangle^n)$  for some set of finitary partial operations  $F$  if and only if

- (a)  $S$  is an algebraic closure system on  $A^n$   
 (b) if  $B \in S$ ,  $1 \leq i < j \leq n$ , then  $B(ij) \in S$ .  
 (c)  $\Delta_2 \times A^{n-2} \in S$   
 (d)  $[C] \{1, 2\} \subseteq [C\{1, 2\}]$  for all nonvoid finite  $C \in P(A^n)$   
 (e) if  $\phi \in S$ , then  $\phi = \bigcap \{B: \phi \neq B \in S\}$ .  
 ( $[C]$  denotes the intersection of all elements of  $S$  containing  $C$ ).

It can be shown that conditions (a), (b), (c), (d) and (e) are independent.

It is clear that  $\Delta_2 \times A^{n-2}$  is a subalgebra of  $\langle A; F \rangle^n$  for all  $F$ . That conditions (a), (b) and (d) are necessary follows from Propositions 1, 2 and 3.

*Proof of Sufficiency.* For every positive integer  $m$  and every ordered  $m + 1$ -tuple  $(a_1, \dots, a_m, a)$  of elements of  $A^n$  such that  $a \in \{[a_1, \dots, a_m]\}$  we associate an  $m$ -ary partial operation  $f$  on  $A$  such that

$Df =$  domain of definition of  $f = \{(a_1(i), \dots, a_m(i)): 1 \leq i \leq n\}$  and

$$a_1(i) \cdots a_m(i) f = a(i), 1 \leq i \leq n.$$

Let  $F$  be the set of all such finitary partial operations.

The following lemmas constitute the proof of sufficiency:

LEMMA 1. If  $C \in P(A^n)$ ,  $s \in S_n$ , then  $[C]s = [Cs]$ .

By (a),  $S$  is a closure system hence  $[C] \in S$ . From (b)  $[C](ij) \in S$  for all  $1 \leq i < j \leq n$ . Hence, by Proposition 2,  $[C]s \in S$  ( $P(A^n) = S(\langle A; \phi \rangle^n)$ ). But

$$Cs \subseteq [C]s \in S.$$

Hence

$$[Cs] \subseteq [C]s.$$

Also

$$C = (Cs)s^{-1}.$$

Hence

$$[C] = [(Cs)s^{-1}] \subseteq [Cs]s^{-1}.$$

And so

$$[C]s \subseteq [Cs].$$

LEMMA 2. If  $\alpha$  is a nonvoid subset of  $\{1, \dots, n\}$  and  $C \in P(A^n)$ ,  $C$  is finite and nonvoid, then

$$[C]\alpha \subseteq [C\alpha].$$

First we show Lemma 2 for the case  $\alpha = \{i, j\}, 1 \leq i < j \leq n$ .

$$\begin{aligned}
 [C] \{i, j\} &= (([C](1i)(2j)) \{1, 2\})(1i)(2j) \\
 &= ([C(1i)(2j)] \{1, 2\})(1i)(2j) \quad \text{(by Lemma 1)} \\
 &\cong [C(1i)(2j) \{1, 2\}](1i)(2j) \quad \text{by (d)} \\
 &= [C(1i)(2j) \{1, 2\}](1i)(2j) \\
 &= [C\{i, j\}].
 \end{aligned}$$

If  $1 \leq i_1 < \dots < i_k \leq n$ , then

$$\begin{aligned}
 [C] \{i_1, \dots, i_k\} &= (\dots([C] \{i_1, i_2\} \{i_2, i_3\}) \dots) \{i_{k-1}, i_k\} \quad \text{(by (3))} \\
 &\cong (\dots([C\{i_1, i_2\}]\{i_2, i_3\}) \dots) \{i_{k-1}, i_k\} \\
 &\cong \dots \\
 &\cong [C\{i_1, \dots, i_k\}].
 \end{aligned}$$

LEMMA 3. *The definition of  $F$  is correct, i.e. every  $f \in F$  is one valued.*

Lemma 3 will be established once we show that whenever  $a_1, \dots, a_m \in A^n, f \in F$  are such that  $a_1(i) \dots a_m(i)f$  is defined for every  $1 \leq i \leq n$  and if for some  $1 \leq p < q \leq n$   $a_1(p) = a_1(q), \dots, a_m(p) = a_m(q)$ ; then

$$a_1(p) \dots a_m(p)f = a_1(q) \dots a_m(q)f.$$

By the definition of  $F$ , there are  $c_1, \dots, c_m, c \in A^n$  such that  $c \in \{c_1, \dots, c_m\}$ ,

$$Df = \{(c_1(i), \dots, c_m(i)): 1 \leq i \leq n\}$$

and

$$c_1(i) \dots c_m(i)f = c(i); 1 \leq i \leq n.$$

Hence

$$\begin{aligned}
 \{(a_1(i), \dots, a_m(i)): 1 \leq i \leq n\} &\subseteq Df \\
 &= \{(c_1(i), \dots, c_m(i)): 1 \leq i \leq n\}.
 \end{aligned}$$

So there are  $s \in S_n$  and  $\alpha$  nonvoid subset of  $\{1, \dots, n\}$  such that

$$a_t = c_t s \alpha, 1 \leq t \leq m.$$

Since every  $a_t$  satisfies  $a_t(p) = a_t(q)$ . We have  $a_t \in (\Delta_2 \times A^{n-2})(1p)(2q) \in S$  (by (c) and (b)) for all  $1 \leq t \leq m$ . Then

$$\{c_1, \dots, c_m\} s \alpha = \{a_1, \dots, a_m\} \subseteq (\Delta_2 \times A^{n-2})(1p)(2q).$$

But

$$\{c_1, \dots, C_m\} s \alpha = \{c_1, \dots, c_m\} s \alpha$$

$$\begin{aligned} &\subseteq \{[c_1, \dots, c_m]s\alpha\} \\ &= \{[a_1, \dots, a_m]\} \\ &\subseteq (\Delta_2 \times A^{n-2})(1p)(2q) . \end{aligned}$$

Define  $a \in A^n$  by

$$a(j) = a_1(j) \cdots a_m(j)f \quad 1 \leq j \leq n .$$

Then

$$a = cs\alpha \in \{[c_1, \dots, c_m]s\alpha\} \subseteq (\Delta_2 \times A^{n-2})(1p)(2q) .$$

Hence

$$a_1(p) \cdots a_m(p)f = a(p) = a(q) = a_1(q) \cdots a_m(q)f .$$

LEMMA 4. *If  $\phi \neq B \in S(\langle A; F \rangle^n)$  then  $B \in S$ .*

Since  $S$  is an algebraic closure system it will be sufficient to show that if  $C$  is a finite nonvoid subset of  $B$ , the  $[C] \subseteq B$ .

Let  $b_1, \dots, b_m \in B$  and  $b \in [b_1, \dots, b_m]$ . By the definition of  $F$ , there is  $f \in F$  such that  $b_1(i) \cdots b_m(i)f$  is defined and is equal to  $b(i)$  for all  $1 \leq i \leq n$ .  $B$  is a subalgebra of  $\langle A; F \rangle^n$ , hence  $b \in B$ .

LEMMA 5. *If  $\phi \neq B \in S$  then  $B \in S(\langle A; F \rangle^n)$ .*

Let  $f \in F$ ;  $a_1, \dots, a_m \in B$  and  $a_1(i) \cdots a_m(i)f = a(i)$ ,  $1 \leq i \leq n$ . We must show that  $a \in B$ .

By the definition of  $F$  there are  $c_1, \dots, c_m, c \in A^n$  such that  $c \in \{[c_1, \dots, c_m]\}$

$$Df = \{(c_1(i), \dots, c_m(i)): 1 \leq i \leq n\}$$

and

$$c_1(i) \cdots c_m(i)f = c(i), 1 \leq i \leq n .$$

So

$$\begin{aligned} &\{(a_1(i), \dots, a_m(i)): 1 \leq i \leq n\} \subseteq Df \\ &= \{(c_1(i), \dots, c_m(i)): 1 \leq i \leq n\} . \end{aligned}$$

As in Lemma 3

$$a_t = c_t s\alpha, 1 \leq t \leq m; a = csd ,$$

for some  $s \in S_n$  and  $\phi \neq a \subseteq \{1, \dots, n\}$ .

But

$$c \in \{[c_1, \dots, c_m]\} .$$

Hence

$$\begin{aligned} a = cs\alpha &\in \{[c_1, \dots, c_m]s\alpha\} \subseteq \{[c_1, \dots, c_m]s\alpha\} \\ &= \{[a_1, \dots, a_m]\} \subseteq B . \end{aligned}$$

THEOREM 5. *Let  $C \subseteq P(A^n)$ .  $C$  is the set of all congruence*

relations on  $\langle A; F \rangle$  for some set of finitary partial operations  $F$  if and only if  $C$  is an algebraic closure system on  $A^2$  and every element of  $C$  is an equivalence relation on  $A$ .

That the set of all congruence relations on  $\langle A; F \rangle$  is an algebraic closure system on  $A^2$  is well known.

If  $C \subseteq P(A^2)$  is a set of equivalence relations which is also an algebraic closure system on  $A^2$  then  $C$  satisfies all the conditions (a), (b), (c), (d) and (e) of the Characterization Theorem. Hence  $C = S(\langle A; F \rangle^2)$  for some set of finitary partial operations  $F$ . Since every element of  $C$  is an equivalence relation on  $A$  and a subalgebra of  $\langle A; F \rangle^2$ , it is a congruence relation on  $\langle A; F \rangle$ . Since a congruence relation on  $\langle A; F \rangle$  is an equivalence relation on  $A$  which is also a subalgebra of  $\langle A; F \rangle^2$ , the Theorem is proved.

6. The following proposition shows that our Characterization Theorem does not solve the corresponding problem for full algebras.

PROPOSITION 4. *There are partial algebras  $\langle A; F \rangle$  such that  $S(\langle A; F \rangle^2) \neq S(\langle A; G \rangle^2)$  for any set of full finitary operations  $G$ .*

Let  $A = \{1, 2, 3\}$ ,  $F = \{f_1, f_2, f_3, g\}$ ;  $f_1, f_2, f_3$  are full unary operations,  $g$  is a partial binary operation.

$f_i$  is the constant function taking the value  $i$ ,  $i = 1, 2, 3$ .

$$\begin{aligned} Dg &= \{(1, 2), (2, 1)\} \\ 12g &= 3, (21)g = 2, \\ B &= \Delta_2 \cup \{(1, 2)\}, C = \Delta_2 \cup \{(2, 1)\} \\ B \circ C &= B \cup C \\ B, C &\in S(\langle A; F \rangle^2), \text{ but} \\ B \circ C &\notin S(\langle A; F \rangle^2) \end{aligned}$$

since any subalgebra of  $\langle A; F \rangle^2$  containing  $(1, 2)$  and  $(2, 1)$  contains also  $(3, 2)$  and  $(2, 3)$ .

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