NORMS OF DERIVATIONS OF $\mathcal{L}(X)$

BARRY E. JOHNSON
NORMS OF DERIVATIONS ON $\mathcal{L}(\mathfrak{K})$

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If $\mathfrak{K}$ is a real or complex Banach space and $\mathcal{L}(\mathfrak{K})$ is the algebra of bounded linear endomorphisms of $\mathfrak{K}$ then each element $T$ of $\mathcal{L}(\mathfrak{K})$ defines an operator $D_T$ on $\mathcal{L}(\mathfrak{K})$ by $D_T(A) = AT - TA$. Clearly $||D_T|| \leq 2 \inf_{\lambda \in K} ||T + \lambda I||$ and Stampfli has shown that when $\mathfrak{K}$ is a complex Hilbert space equality holds. In this paper it is shown, by methods which apply to a large class of uniformly convex spaces, that this formula for $||D_T||$ is false in $l^p$ and $L^p(0,1)$ $1 < p < \infty$, $p \neq 2$. For $L^1$ spaces the formula is true in the real case but not in the complex case when the space has dimension 3 or more.

Stampfli's results appear in [1] stated for complex Hilbert space but the same proofs yield the corresponding result for real spaces.

Throughout this paper $K$ will denote either $\mathbb{R}$ or $\mathbb{C}$. We begin by describing the construction of an operator $T$ of rank 1 with $||D_T|| < d_T = 2 \inf_{\lambda \in K} ||T + \lambda I||$. The reason that this fails in Hilbert space is precisely because for an ellipse, conjugacy is a symmetric relation on the set of diameters; more precisely if $x, y$ are two points on the unit ball then $y$ is parallel to the tangent plane at $x$ if and only if $x$ is parallel to the tangent plane at $y$.

**Definition 1.** Let $x \in \mathfrak{K}$, $||x|| = 1$. The unit ball $\mathfrak{K}_x$ is uniformly convex at $x$ if whenever $\{y_n\}$ is a sequence with $||y_n|| \leq 1$, $||x + y_n|| \to 2$ then $y_n \to x$.

**Proposition 2.** Let $\mathfrak{K}$ be a normed space over $K$ and let $x, y \in \mathfrak{K}$ with the following properties

( i ) $||x|| = 1$ and there is $f \in \mathfrak{K}^*$ with $||f|| = 1$ and such that if $\{x_n\}$ is a sequence with $||x_n|| \leq 1$, $f(x_n) \to 1$ then $x_n \to x$.

( ii ) $||y|| = 1$ and the unit ball $\mathfrak{K}_y$ is uniformly convex at $y$.

( iii ) For some $\lambda \in K$, $||x + \lambda y|| < 1$.

( iv ) For all $\lambda$ in $K$, $||y + \lambda x|| \geq 1$.

Define $T \in \mathcal{L}(\mathfrak{K})$ by $Tz = f(z)y$. Then $2||T|| = d_T > ||D_T||$.

**Proof.** $||T + \lambda I|| \geq ||(T + \lambda I)x|| = ||y + \lambda x|| \geq ||y|| = 1$ by (iv) and $||T|| = 1$ so $d_T = 2$. Suppose $||D_T|| = 2$ and choose sequences $\{A_n\}$ from $\mathcal{L}(\mathfrak{K})$ and $\{x_n\}$ from $\mathfrak{K}$ with $||A_n|| = 1 = ||x_n||$ and $||D_T(A_n)x_n|| \to 2$. As $||TA_nx_n|| \leq 1$, $||A_nTx_n|| \leq 1$ we have $||TA_nx_n|| \to 1$, $||A_nTx_n|| \to 1$. 

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and hence \(|A_n x_n| \to 1, |T x_n| \to 1\). This shows \(|f(x_n)| \to 1\) and so, replacing \(x_n\) by \(w_n x_n\) if necessary, where \(\{w_n\}\) is a sequence of elements of \(K\) with \(|w_n| = 1\), we may assume \(f(x_n) \to 1\). Condition (i) now implies \(x_n \to x\) and hence \(T x_n \to y\). In the same way \(|A_n x_n| \to 1\) implies \(|f(A_n x_n)| \to 1\) and replacing \(A_n\) by \(w_n A_n\) if necessary, we can assume \(f(A_n x_n) \to 1\) from which we see \(A_n x_n \to x, TA_n x_n \to y\). As \(|A_n| \leq 1\) we have \(A_n T x_n - A_n y \to 0\) and so \(|A_n T x_n - TA_n x_n| \to 2\) implies \(|A_n y - y| \to 2\). Condition (ii) now shows \(A_n y \to -y\) so that \(A_n(x + \lambda y) \to x - \lambda y\). However if \(\lambda\) satisfies condition (iii) then \(|x - \lambda y| > 1\), as otherwise \(2 = 2 ||x|| \leq ||x + \lambda y|| + ||x - \lambda y|| < 2\), and so \(\lim |A_n(x + \lambda y)| = ||x - \lambda y|| > 1\) which is impossible because \(|A_n(x + \lambda y)| \leq ||A_n|| ||x + \lambda y|| < 1\).

**Proposition 3.** Let \(\mathfrak{F}\) be a uniformly convex Banach space, \(x, y \in \mathfrak{F}, f, g \in \mathfrak{F}^*\) with \(||x|| = ||y|| = ||f|| = ||g|| = f(x) = g(y) = 1, g(x) = 0, f(y) \neq 0\) and suppose \(f\) is the only element \(h\) of \(\mathfrak{F}^*\) with \(||h|| = h(x) = 1\). Then \(x, y, f\) satisfy the conditions of Proposition 2.

**Proof.** (i) If \(||x_n|| \leq 1, f(x_n) \to 1\) then \(f(x + x_n) \to 2\) and as \(||x + x_n|| \leq 2, ||f|| = 1\) we have \(||x + x_n|| \to 2\) so \(x_n \to x\) by uniform convexity.

(ii) is clearly part of the present hypotheses.

(iii) \(x\) and \(y\) are linearly independent as \(g(x) = 0, g(y) = 1, x \neq 0\). If \(||x + \lambda y|| \geq 1\) for all \(\lambda \in K\) then \(\alpha x + \beta y \mapsto \alpha\) is a norm one linear functional on the space spanned by \(x\) and \(y\) and so has an extension \(h\) in \(\mathfrak{F}^*\) with \(||h|| = 1, h(x) = 1\) but \(h \neq f\) because \(h(y) = 0 \neq f(y)\).

(iv) As \(g(y + \lambda x) = g(y) = 1\), for all \(\lambda\) in \(K\) and \(||g|| = 1\) we have \(||y + \lambda x|| \geq 1\) for all \(\lambda\) in \(K\).

**Corollary 4.** If \(1 < p < 2\) or \(2 < p < \infty\) and \(\mathfrak{F} = l^p(0, \infty)\) or \(\mathfrak{F} = L^p(-1, +1)\) is the corresponding \(K\) Banach space of \(K\) valued functions then there is \(T \in \mathcal{L}(\mathfrak{F})\) with \(||D_T|| \neq d_T\).

**Proof.** The spaces are uniformly convex and at each point \(z\) of \(\mathfrak{F}\) with \(||z|| = 1\) the element \(h\) of \(\mathfrak{F}^*\) with \(h(z) = 1 = ||h||\) is unique. Thus the construction in Proposition 2 applies once we find two suitable points \(x, y\) and these exist in such abundance that we can take anything but multiples of characteristic functions for \(x\). First of all we give the construction in the two dimensional space \(l^p(1,2)\).

If \(x = (x_1, x_2), x_1 > 0, x_2 > 0, x_1^p + x_2^p = 1\) then \(f(z) = x_1^{p-1} z_1 + x_2^{p-1} z_2\) so \(y\) can be taken as \(\alpha(x_2^{p-1}, -x_1^{p-1})\) where \(\alpha^{-p} = x_1^{p(p-1)} + x_2^{p(p-1)}\) and
\( g(z) = \alpha^{x_1-1}(x_2^{x_2-1})^2 z_1 - \alpha^{x_2-1}(x_1^{x_1-1})^2 z_2 \). Then \( g(x) = \alpha^{x_1-1}(x_1^{x_2^{x_2-1}} - x_2^{x_1^{x_1-1}}) \), which will be zero if and only if \( x_1 = x_2 \). Thus taking say \( x = 3^{1/p}(2^{1/p}, 1) \) and \( y, \alpha, g \) as above the result is shown in \( l^p(1, 2) \).

As \( l^p(0, \infty) \) and \( L^p(-1, +1) \) each contain subspaces isometric with \( l^p(1, 2) \) we can construct \( x, y, f, g \) in this subspace and then extend \( f \) and \( g \) to \( \mathfrak{x} \) using the Hahn-Banach theorem.

In order to prove the results for spaces of measures we establish the equation \( d_r = ||D_r|| \) for finite dimensional \( l^r \) spaces.

**Proposition 5.** Let \( n \) be a positive integer and \( \mathfrak{x} \) be the real Banach space \( R^n \) with norm \( ||x|| = \Sigma |x_i| \). Let \( T \in \mathfrak{L}(\mathfrak{x}) \). Then \( ||D_r|| = 2 \inf_{x \in R} ||T + \lambda I|| \).

**Proof.** Suppose \( T \) is given by the matrix \( a_{ij} \) in the standard basis \( e_1, e_2, \ldots, e_n \). We have \( ||T|| = \sup_j \Sigma_i |a_{ij}| \). Suppose \( \Sigma_i |a_{ij}| = ||T|| \) for \( j = 1, \ldots, m \) but not for \( j > m \). The condition \( ||T|| = \frac{1}{2} d_r \) is equivalent to saying that 0 is in the convex hull of \( a_{11}, \ldots, a_{mm} \) since if 0 does not lie in this convex hull then either \( |a_{jj} + \lambda| < |a_{jj}| \) for \( j = 1, \ldots, m \) and small positive \( \lambda \) or for small negative \( \lambda \) and so there are small values of \( \lambda \) with \( ||T + \lambda I|| < ||T|| \) whereas if 0 lies in this hull and \( \lambda \neq 0 \) there is \( j \) with \( 1 \leq j \leq m \) and \( |a_{jj} + \lambda| > |a_{jj}| \) so that \( ||T + \lambda I|| > ||T|| \).

It is clearly sufficient to prove the result when \( ||T|| = \frac{1}{2} d_r \). First of all consider the case \( m \geq 2 \) and suppose \( a_{11} \geq 0 \geq a_{22} \). Let \( A \in \mathfrak{L}(\mathfrak{x}) \) be an operator of the form \( Ae_1 = e_2, Ae_2 = \pm e_1, Ae_i = \pm e_i \) \( i = 3, \ldots, n \). Clearly \( ||A|| = 1 \) and

\[
||D_r(A)e_1|| = ||ATE_e - Te_2|| \\
= |\pm a_{21} - a_{12}| + |a_{11} - a_{22}| + \sum_{i=3}^{n} |\pm a_{i1} - a_{i2}| \\
= \sum_{i=1}^{n} |a_{i1}| + \sum_{i=1}^{n} |a_{i2}| \\
= 2 ||T||
\]

for a suitable choice of signs of the \( Ae_i \) since each sign to be chosen corresponds to exactly one term \( \pm a_{i1} - a_{i2} \).

If \( m = 1 \) then \( a_{11} = 0 \) because 0 lies in the convex hull of \( a_{11}, \ldots, a_{mm} \), and we define \( A \) by \( Ae_1 = e_1, Ae_j = -e_j \) \( j = 2, \ldots, n \) which gives \( ||A|| = 1 \) and \( ATe_1 = -Te_1 \) so that

\[
||D_r(A)e_1|| = ||ATE_e - T Ae_e|| = 2 ||Te_1|| = 2 ||T|| .
\]

**Proposition 6.** Let \( \Omega \) be a compact topological space and \( \mathfrak{x} \) a closed linear subspace of the (real) Banach space of real valued measures on \( \Omega \) with the property that if \( \mu \in \mathfrak{x} \) then every measure
absolutely continuous with respect to \( \mu \) is in \( \mathcal{X} \). Let \( T \in \mathcal{L}(\mathcal{X}) \). Then
\[
\|D_T\| = 2 \inf_{\lambda \in \mathbb{R}} \|T + \lambda I\|.
\]

**Proof.** We may assume \( d_T = 2 \|T\| \). Let \( \varepsilon > 0 \). For each \( \nu > 0 \) in \( \mathcal{X} \) let \( E_\nu(\mu) \) be the part of \( \mu \in \mathcal{X} \) which is absolutely continuous with respect to \( \nu \). The \( E_\nu \) form a system of commuting idempotents of norm 1 and \( E_\nu E_{\nu'} = E_\nu \) if \( \nu' > \nu \), so that \( \|E_\nu S E_\nu\| \), where the elements \( \nu \) are directed by the usual ordering of measures, is a monotonic direct net. It is easy to see that \( \|E_\nu S E_\nu\| \to \|S\| \). Thus applying Dini's theorem to the functions \( \lambda \mapsto \|E_\nu(T + \lambda I)E_\nu\| \) we can find \( \nu \in \mathcal{X} \), \( \nu > 0 \) with \( \|E_\nu(T + \lambda I)E_\nu\| > \|T + \lambda I\| - \varepsilon \) for all \( |\lambda| \leq 2 \|T\| \).

For each dissection \( \Delta = (\Omega_1, \ldots, \Omega_n) \) of \( \Omega \) into disjoint measurable sets of positive \( \nu \) measure we define
\[
P_\Delta(\mu) = (E_\nu(\mu)(\Omega_1), \ldots, E_\nu(\mu)(\Omega_n))
\]
\[
Q_\Delta(\xi) = (\sum c_i \xi_i \nu(\Omega_i)^-i)\nu
\]
where \( \mu \in \mathcal{X}, \xi \in \mathbb{R}^n, P_\Delta : \mathbb{R}^n \to \mathbb{R}^n, Q_\Delta : \mathbb{R}^n \to \mathcal{X} \) and \( c_i \) is the characteristic function of \( \Omega_i \). Directing the dissections in the usual way it is easy to see that for each \( S \in \mathcal{L}(\mathcal{X}) \) \( \|P_\Delta E_\nu S E_\nu Q_\Delta\| \), where \( \mathbb{R}^n \) has the \( l^1 \) norm, is a monotonic directed set with limit \( \|E_\nu S E_\nu\| \). Applying Dini's theorem again we see that there is a dissection \( \Delta \) with
\[
(*) \quad \|P_\Delta E_\nu(T + \lambda I)E_\nu Q_\Delta\| > \|T\| - \varepsilon
\]
for all \( |\lambda| \leq 2 \|T\| \). For convenience we now denote \( E_\nu, P_\Delta, Q_\Delta \) by \( E, P, Q \). As these operators have norm 1 we see that inequality (*) holds for all values of \( \lambda \). As \( PE = P, EQ = Q, PEQ = PQ = \text{identity on} \mathbb{R}^n \), (*) shows that \( d_{PQ} \geq 2(\|T\| - \varepsilon) \). By proposition 5 there is \( A \in \mathcal{L}(\mathbb{R}^n) \) with \( \|D_{PQ}(A)\| = d_{PQ}, \|A\| = 1 \). As \( Q \) is an isometry and \( P \) maps the unit ball of \( \mathbb{X} \) onto that of \( \mathbb{R}^n \) we have
\[
d_{PQ} = \|QD_{PQ}(A)P\|
= \|QAPTQP - QPTQAP\|
= \|QPDP_T(QAP)QP\|
\leq \|D_T(QAP)\|.
\]
As \( \|QAP\| = 1 \) we have \( \|D_T\| \geq d_{PQ} \geq 2(\|T\| - \varepsilon) \) for each \( \varepsilon > 0 \) and the result follows.

In the complex space \( l^1(1,2) \) Proposition 5 is true and the proof is similar to that for the real case. However the result is false in higher dimensions for complex spaces, e.g., in \( l^1(1,2,3) \) let \( T \) be the linear transformation given by the matrix
where \( \omega^3 = 1, \omega \neq 1 \). The situation is similar to that at the beginning of the proof of Proposition 5 with \( m = n = 3 \) and the argument given there shows that because 0 is a convex combination of diagonal entries we have \( \inf_{x \in c} \| T + \lambda I \| = \| T \| = 3 \). If \( \| x \| = 1, \| A \| = 1 \) and \( \| D_r(A)x \| = 6 \) then \( \| Tx \| = 3 \) and since \( |x_1 \pm \omega x_2 \pm \omega^2 x_3| \leq 1 \) we see that

\[
|x_1 - \omega x_2 - \omega^2 x_3| = |x_1 + \omega x_2 - \omega^2 x_3| = |x_1 + \omega x_2 + \omega^2 x_3| = |x_1| + |x_2| + |x_3|
\]

which occurs only if two of \( x_1, x_2, x_3 \) are 0. Multipling by a complex number of absolute value 1, if necessary, we can assume \( x = e_1 \) or \( e_2 \) or \( e_3 \). In the same way \( Ax = e_1 \) or \( e_2 \) or \( e_3 \). If \( x = e_1 = Ax \) then

\[
\| D_r(A)e_1 \| = \| e_1 + Ae_2 + Ae_3 - e_1 - e_2 - e_3 \|
= \| Ae_2 + Ae_3 - e_2 - e_3 \| \\
\leq 4
\]

and if \( x = e_1, Ax = e_2 \) then

\[
\| D_r(A)e_2 \| = \| e_2 + Ae_2 + Ae_3 + \omega e_1 - \omega e_2 - \omega e_3 \|
= \| (1 - \omega)e_2 + Ae_2 + Ae_3 - \omega e_1 - \omega e_3 \| \\
\leq \sqrt{3} + 4.
\]

The other four possibilities give similar results and so we cannot in fact have \( \| D_r \| = 6 \).

A similar construction in the complex spaces \( l'(1, n), l'(0, \infty), L'(0, 1), M(0, 1) \) shows that Proposition 6 is false in these spaces too.

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