

# Pacific Journal of Mathematics

**NORMS OF DERIVATIONS OF  $\mathcal{L}(X)$**

BARRY E. JOHNSON

## NORMS OF DERIVATIONS ON $\mathcal{L}(\mathfrak{X})$

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If  $\mathfrak{X}$  is a real or complex Banach space and  $\mathcal{L}(\mathfrak{X})$  is the algebra of bounded linear endomorphisms of  $\mathfrak{X}$  then each element  $T$  of  $\mathcal{L}(\mathfrak{X})$  defines an operator  $D_T$  on  $\mathcal{L}(\mathfrak{X})$  by  $D_T(A) = AT - TA$ . Clearly  $\|D_T\| \leq 2 \inf_{\lambda} \|T + \lambda I\|$  and Stampfli has shown that when  $\mathfrak{X}$  is a complex Hilbert space equality holds. In this paper it is shown, by methods which apply to a large class of uniformly convex spaces, that this formula for  $\|D_T\|$  is false in  $l^p$  and  $L^p(0, 1)$   $1 < p < \infty$ ,  $p \neq 2$ . For  $L^1$  spaces the formula is true in the real case but not in the complex case when the space has dimension 3 or more.

Stampfli's results appear in [1] stated for complex Hilbert space but the same proofs yield the corresponding result for real spaces.

Throughout this paper  $K$  will denote either  $R$  or  $C$ . We begin by describing the construction of an operator  $T$  of rank 1 with  $\|D_T\| < d_T = 2 \inf_{\lambda \in K} \|T + \lambda I\|$ . The reason that this fails in Hilbert space is precisely because for an ellipse, conjugacy is a symmetric relation on the set of diameters; more precisely if  $x, y$  are two points on the unit ball then  $y$  is parallel to the tangent plane at  $x$  if and only if  $x$  is parallel to the tangent plane at  $y$ .

**DEFINITION 1.** Let  $x \in \mathfrak{X}$ ,  $\|x\| = 1$ . The unit ball  $\mathfrak{X}_1$  is *uniformly convex* at  $x$  if whenever  $\{y_n\}$  is a sequence with  $\|y_n\| \leq 1$ ,  $\|x + y_n\| \rightarrow 2$  then  $y_n \rightarrow x$ .

**PROPOSITION 2.** Let  $\mathfrak{X}$  be a normed space over  $K$  and let  $x, y \in \mathfrak{X}$  with the following properties

- (i)  $\|x\| = 1$  and there is  $f \in \mathfrak{X}^*$  with  $\|f\| = 1$  and such that if  $\{x_n\}$  is a sequence with  $\|x_n\| \leq 1$ ,  $f(x_n) \rightarrow 1$  then  $x_n \rightarrow x$ .
- (ii)  $\|y\| = 1$  and the unit ball  $\mathfrak{X}_1$  is uniformly convex at  $y$ .
- (iii) For some  $\lambda \in K$ ,  $\|x + \lambda y\| < 1$ .
- (iv) For all  $\lambda$  in  $K$ ,  $\|y + \lambda x\| \geq 1$ .

Define  $T \in \mathcal{L}(\mathfrak{X})$  by  $Tz = f(z)y$ . Then  $2\|T\| = d_T > \|D_T\|$ .

*Proof.*  $\|T + \lambda I\| \geq \|(T + \lambda I)x\| = \|y + \lambda x\| \geq \|y\| = 1$  by (iv) and  $\|T\| = 1$  so  $d_T = 2$ . Suppose  $\|D_T\| = 2$  and choose sequences  $\{A_n\}$  from  $\mathcal{L}(\mathfrak{X})$  and  $\{x_n\}$  from  $\mathfrak{X}$  with  $\|A_n\| = 1 = \|x_n\|$  and  $\|D_T(A_n)x_n\| \rightarrow 2$ . As  $\|TA_nx_n\| \leq 1$ ,  $\|A_nTx_n\| \leq 1$  we have  $\|TA_nx_n\| \rightarrow 1$ ,  $\|A_nTx_n\| \rightarrow 1$

and hence  $\|A_n x_n\| \rightarrow 1, \|Tx_n\| \rightarrow 1$ . This shows  $|f(x_n)| \rightarrow 1$  and so, replacing  $x_n$  by  $w_n x_n$  if necessary where  $\{w_n\}$  is a sequence of elements of  $K$  with  $|w_n| = 1$ , we may assume  $f(x_n) \rightarrow 1$ . Condition (i) now implies  $x_n \rightarrow x$  and hence  $Tx_n \rightarrow y$ . In the same way  $\|TA_n x_n\| \rightarrow 1$  implies  $|f(A_n x_n)| \rightarrow 1$  and replacing  $A_n$  by  $w'_n A_n$  if necessary we can assume  $f(A_n x_n) \rightarrow 1$  from which we see  $A_n x_n \rightarrow x, TA_n x_n \rightarrow y$ . As  $\|A_n\| \leq 1$  we have  $A_n Tx_n - A_n y \rightarrow 0$  and so  $\|A_n Tx_n - TA_n x_n\| \rightarrow 2$  implies  $\|A_n y - y\| \rightarrow 2$ . Condition (ii) now shows  $A_n y \rightarrow -y$  so that  $A_n(x + \lambda y) \rightarrow x - \lambda y$ . However if  $\lambda$  satisfies condition (iii) then  $\|x - \lambda y\| > 1$ , as otherwise  $2 = 2\|x\| \leq \|x + \lambda y\| + \|x - \lambda y\| < 2$ , and so  $\lim \|A_n(x + \lambda y)\| = \|x - \lambda y\| > 1$  which is impossible because  $\|A_n(x + \lambda y)\| \leq \|A_n\| \|x + \lambda y\| < 1$ .

**PROPOSITION 3.** *Let  $\mathfrak{X}$  be a uniformly convex Banach space,  $x, y \in \mathfrak{X}, f, g \in \mathfrak{X}^*$  with  $\|x\| = \|y\| = \|f\| = \|g\| = f(x) = g(y) = 1, g(x) = 0, f(y) \neq 0$  and suppose  $f$  is the only element  $h$  of  $\mathfrak{X}^*$  with  $\|h\| = h(x) = 1$ . Then  $x, y, f$  satisfy the conditions of Proposition 2.*

*Proof.* (i) If  $\|x_n\| \leq 1, f(x_n) \rightarrow 1$  then  $f(x + x_n) \rightarrow 2$  and as  $\|x + x_n\| \leq 2, \|f\| = 1$  we have  $\|x + x_n\| \rightarrow 2$  so  $x_n \rightarrow x$  by uniform convexity.

(ii) is clearly part of the present hypotheses.

(iii)  $x$  and  $y$  are linearly independent as  $g(x) = 0, g(y) = 1, x \neq 0$ . If  $\|x + \lambda y\| \geq 1$  for all  $\lambda \in K$  then  $\alpha x + \beta y \mapsto \alpha$  is a norm one linear functional on the space spanned by  $x$  and  $y$  and so has an extension  $h$  in  $\mathfrak{X}^*$  with  $\|h\| = 1, h(x) = 1$  but  $h \neq f$  because  $h(y) = 0 \neq f(y)$ .

(iv) As  $g(y + \lambda x) = g(y) = 1$ , for all  $\lambda$  in  $K$  and  $\|g\| = 1$  we have  $\|y + \lambda x\| \geq 1$  for all  $\lambda$  in  $K$ .

**COROLLARY 4.** *If  $1 < p < 2$  or  $2 < p < \infty$  and  $\mathfrak{X} = l^p(0, \infty)$  or  $\mathfrak{X} = L^p(-1, +1)$  is the corresponding  $K$  Banach space of  $K$  valued functions then there is  $T \in \mathcal{L}(\mathfrak{X})$  with  $\|D_T\| \neq d_T$ .*

*Proof.* The spaces are uniformly convex and at each point  $z$  of  $\mathfrak{X}$  with  $\|z\| = 1$  the element  $h$  of  $\mathfrak{X}^*$  with  $h(z) = 1 = \|h\|$  is unique. Thus the construction in Proposition 2 applies once we find two suitable points  $x, y$  and these exist in such abundance that we can take anything but multiples of characteristic functions for  $x$ . First of all we give the construction in the two dimensional space  $l^p(1, 2)$ .

If  $x = (x_1, x_2), x_1 > 0, x_2 > 0, x_1^p + x_2^p = 1$  then  $f(z) = x_1^{p-1}z_1 + x_2^{p-1}z_2$  so  $y$  can be taken as  $\alpha(x_2^{p-1}, -x_1^{p-1})$  where  $\alpha^{-p} = x_1^{p(p-1)} + x_2^{p(p-1)}$  and

$g(z) = \alpha^{p-1}(x_2^{(p-1)^2} z_1 - x_1^{(p-1)^2} z_2)$ . Then  $g(x) = \alpha^{p-1}(x_1 x_2^{(p-1)^2} - x_1^{(p-1)^2} x_2)$  which will be zero if and only if  $x_1 = x_2$ . Thus taking say  $x = 3^{-1/p}(2^{1/p}, 1)$  and  $y, f, g$  as above the result is shown in  $l^p(1, 2)$ .

As  $l^p(0, \infty)$  and  $L^p(-1, +1)$  each contain subspaces isometric with  $l^p(1, 2)$  we can construct  $x, y, f, g$  in this subspace and then extend  $f$  and  $g$  to  $\mathfrak{X}$  using the Hahn-Banach theorem.

In order to prove the results for spaces of measures we establish the equation  $d_T = \|D_T\|$  for finite dimensional  $l^p$  spaces.

**PROPOSITION 5.** *Let  $n$  be a positive integer and  $\mathfrak{X}$  be the real Banach space  $\mathbf{R}^n$  with norm  $\|x\| = \sum |x_i|$ . Let  $T \in \mathcal{L}(\mathfrak{X})$ . Then  $\|D_T\| = 2 \inf_{\lambda \in \mathbf{R}} \|T + \lambda I\|$ .*

*Proof.* Suppose  $T$  is given by the matrix  $a_{ij}$  in the standard basis  $e_1, e_2, \dots, e_n$ . We have  $\|T\| = \sup_j \sum_i |a_{ij}|$ . Suppose  $\sum_i |a_{ij}| = \|T\|$  for  $j = 1, \dots, m$  but not for  $j > m$ . The condition  $\|T\| = \frac{1}{2} d_T$  is equivalent to saying that 0 is in the convex hull of  $a_{11}, \dots, a_{mm}$  since if 0 does not lie in this convex hull then either  $|a_{jj} + \lambda| < |a_{jj}|$  for  $j = 1, \dots, m$  and small positive  $\lambda$  or for small negative  $\lambda$  and so there are small values of  $\lambda$  with  $\|T + \lambda I\| < \|T\|$  whereas if 0 lies in this hull and  $\lambda \neq 0$  there is  $j$  with  $1 \leq j \leq m$  and  $|a_{jj} + \lambda| > |a_{jj}|$  so that  $\|T + \lambda I\| > \|T\|$ .

It is clearly sufficient to prove the result when  $\|T\| = \frac{1}{2} d_T$ . First of all consider the case  $m \geq 2$  and suppose  $a_{11} \geq 0 \geq a_{22}$ . Let  $A \in \mathcal{L}(\mathfrak{X})$  be an operator of the form  $Ae_1 = e_2, Ae_2 = \pm e_1, Ae_i = \pm e_i, i = 3, \dots, n$ . Clearly  $\|A\| = 1$  and

$$\begin{aligned} \|D_T(A)e_1\| &= \|ATe_1 - Te_2\| \\ &= |\pm a_{21} - a_{12}| + |a_{11} - a_{22}| + \sum_{i=3}^n |\pm a_{i1} - a_{i2}| \\ &= \sum_{i=1}^n |a_{i1}| + \sum_{i=1}^n |a_{i2}| \\ &= 2 \|T\| \end{aligned}$$

for a suitable choice of signs of the  $Ae_i$  since each sign to be chosen corresponds to exactly one term  $|\pm a_{i1} - a_{i2}|$ .

If  $m = 1$  then  $a_{11} = 0$  because 0 lies in the convex hull of  $a_{11}, \dots, a_{mm}$ , and we define  $A$  by  $Ae_1 = e_1, Ae_j = -e_j, j = 2, \dots, n$  which gives  $\|A\| = 1$  and  $ATe_1 = -Te_1$  so that

$$\|D_T(A)e_1\| = \|ATe_1 - TAe_1\| = 2 \|Te_1\| = 2 \|T\|.$$

**PROPOSITION 6.** *Let  $\Omega$  be a compact topological space and  $\mathfrak{X}$  a closed linear subspace of the (real) Banach space of real valued measures on  $\Omega$  with the property that if  $\mu \in \mathfrak{X}$  then every measure*

absolutely continuous with respect to  $\mu$  is in  $\mathfrak{X}$ . Let  $T \in \mathcal{L}(\mathfrak{X})$ . Then  $\|D_T\| = 2 \inf_{\lambda \in \mathbf{R}} \|T + \lambda I\|$ .

*Proof.* We may assume  $d_T = 2 \|T\|$ . Let  $\varepsilon > 0$ . For each  $\nu > 0$  in  $\mathfrak{X}$  let  $E_\nu(\mu)$  be the part of  $\mu \in \mathfrak{X}$  which is absolutely continuous with respect to  $\nu$ . The  $E_\nu$  form a system of commuting idempotents of norm 1 and  $E_\nu E_{\nu'} = E_\nu$  if  $\nu' > \nu$ , so that  $\|E_\nu S E_\nu\|$ , where the elements  $\nu$  are directed by the usual ordering of measures, is a monotonic direct net. It is easy to see that  $\|E_\nu S E_\nu\| \rightarrow \|S\|$ . Thus applying Dini's theorem to the functions  $\lambda \mapsto \|E_\nu(T + \lambda I)E_\nu\|$  we can find  $\nu \in \mathfrak{X}, \nu > 0$  with  $\|E_\nu(T + \lambda I)E_\nu\| > \|T + \lambda I\| - \varepsilon \geq \|T\| - \varepsilon$  for  $|\lambda| \leq 2 \|T\|$ .

For each dissection  $\mathcal{A} = (\Omega_1, \dots, \Omega_n)$  of  $\Omega$  into disjoint measurable sets of positive  $\nu$  measure we define

$$P_{\mathcal{A}}(\mu) = (E_\nu \mu(\Omega_1), \dots, E_\nu \mu(\Omega_n))$$

$$Q_{\mathcal{A}}(\xi) = (\sum c_i \xi_i \nu(\Omega_i)^{-1}) \nu$$

where  $\mu \in \mathfrak{X}, \xi \in \mathbf{R}^n, P_{\mathcal{A}}: \mathfrak{X} \rightarrow \mathbf{R}^n, Q_{\mathcal{A}}: \mathbf{R}^n \rightarrow \mathfrak{X}$  and  $c_i$  is the characteristic function of  $\Omega_i$ . Directing the dissections in the usual way it is easy to see that for each  $S \in \mathcal{L}(\mathfrak{X})$   $\|P_{\mathcal{A}} E_\nu S E_\nu Q_{\mathcal{A}}\|$ , where  $\mathbf{R}^n$  has the  $l^1$  norm, is a monotonic directed set with limit  $\|E_\nu S E_\nu\|$ . Applying Dini's theorem again we see that there is a dissection  $\mathcal{A}$  with

$$(*) \quad \|P_{\mathcal{A}} E_\nu(T + \lambda I)E_\nu Q_{\mathcal{A}}\| > \|T\| - \varepsilon$$

for all  $|\lambda| \leq 2 \|T\|$ . For convenience we now denote  $E_\nu, P_{\mathcal{A}}, Q_{\mathcal{A}}$  by  $E, P, Q$ . As these operators have norm 1 we see that inequality (\*) holds for all values of  $\lambda$ . As  $PE = P, EQ = Q, PEQ = PQ =$  identity on  $\mathbf{R}^n$ , (\*) shows that  $d_{PTQ} \geq 2(\|T\| - \varepsilon)$ . By proposition 5 there is  $A \in \mathcal{L}(\mathbf{R}^n)$  with  $\|D_{PTQ}(A)\| = d_{PTQ}, \|A\| = 1$ . As  $Q$  is an isometry and  $P$  maps the unit ball of  $\mathfrak{X}$  onto that of  $\mathbf{R}^n$  we have

$$\begin{aligned} d_{PTQ} &= \|QD_{PTQ}(A)P\| \\ &= \|QAPTQP - QPTQAP\| \\ &= \|QPD_T(QAP)QP\| \\ &\leq \|D_T(QAP)\|. \end{aligned}$$

As  $\|QAP\| = 1$  we have  $\|D_T\| \geq d_{PTQ} \geq 2(\|T\| - \varepsilon)$  for each  $\varepsilon > 0$  and the result follows.

In the complex space  $l(1, 2)$  Proposition 5 is true and the proof is similar to that for the real case. However the result is false in higher dimensions for complex spaces, e.g., in  $l(1, 2, 3)$  let  $T$  be the linear transformation given by the matrix

$$\begin{matrix} 1 & -\omega & -\omega^2 \\ 1 & \omega & -\omega^2 \\ 1 & \omega & \omega^2 \end{matrix}$$

where  $\omega^3 = 1, \omega \neq 1$ . The situation is similar to that at the beginning of the proof of Proposition 5 with  $m = n = 3$  and the argument given there shows that because  $0$  is a convex combination of diagonal entries we have  $\inf_{\lambda \in \mathbb{C}} \|T + \lambda I\| = \|T\| = 3$ . If  $\|x\| = 1, \|A\| = 1$  and  $\|D_T(A)x\| = 6$  then  $\|Tx\| = 3$  and since  $|x_1 \pm \omega x_2 \pm \omega^2 x_3| \leq 1$  we see that  $|x_1 - \omega x_2 - \omega^2 x_3| = |x_1 + \omega x_2 - \omega^2 x_3| = |x_1 + \omega x_2 + \omega^2 x_3| = |x_1| + |x_2| + |x_3|$  which occurs only if two of  $x_1, x_2, x_3$  are  $0$ . Multiplying by a complex number of absolute value  $1$ , if necessary, we can assume  $x = e_1$  or  $e_2$  or  $e_3$ . In the same way  $Ax = e_1$  or  $e_2$  or  $e_3$ . If  $x = e_1 = Ax$  then

$$\begin{aligned} \|D_T(A)e_1\| &= \|e_1 + Ae_2 + Ae_3 - e_1 - e_2 - e_3\| \\ &= \|Ae_2 + Ae_3 - e_2 - e_3\| \\ &\leq 4 \end{aligned}$$

and if  $x = e_1, Ax = e_2$  then

$$\begin{aligned} \|D_T(A)e_1\| &= \|e_2 + Ae_2 + Ae_3 + \omega e_1 - \omega e_2 - \omega e_3\| \\ &= \|(1 - \omega)e_2 + Ae_2 + Ae_3 - \omega e_1 - \omega e_3\| \\ &\leq \sqrt{3} + 4. \end{aligned}$$

The other four possibilities give similar results and so we cannot in fact have  $\|D_T\| = 6$ .

A similar construction in the complex spaces  $l^1(1, n), l^1(0, \infty), L^1(0, 1), M(0, 1)$  shows that Proposition 6 is false in these spaces too.

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