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In this paper we solve the triple integral equations

$$(1)\qquad \mathfrak{M}^{-1}\left\{\frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\,\Phi(s);\ x\right\}=0,\ 0\leqq x<a,\ b<x<\infty\ ,$$

$$(2)\qquad \mathfrak{M}^{-1}\left\{\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\,\Phi(s);\ x\right\}=f_2(x),\ a<x<b\ ,$$

where $\alpha,\beta,\xi,\eta,\delta>0,\ \sigma>0$, **are real parameters,** $f_2(x)$ **is a known function,** $\Phi(s)$ **is to be determined and**

$$(3)\qquad \mathfrak{M}\{h(x);\ s\}=H(s),\ \mathfrak{M}^{-1}\{H(s);\ x\}=h(x)\ ,$$

denote the Mellin transform of $h(x)$ **and its inversion formula respectively.**

The above equations are an extension of the dual integral equations solved in a recent paper by Erdélyi [2] by means of a systematic application of the Erdélyi-Kober operators of fractional integration [4].

Using the properties of some slightly extended forms of the Erdélyi-Kober operators we show, in a purely formal manner, that the solution of the triple integral equations can be expressed in terms of the solution of a Fredholm integral equation of the second kind. Srivastav and Parihar [5] have solved a very special case of the equations by a completely different method from that used in this paper. The method of solution employed here will be seen to follow closely that used by Cooke [1] to obtain the solution to some triple integral equations involving Bessel functions; indeed Cooke's equations may be regarded as a special case of equations (1) and (2) and it is shown that a solution of his equations can be readily obtained from that presented in this paper.

2. **The integral operators.** We shall use the integral operators defined by

$$(4)\qquad I_{\eta,\alpha}(a,x:\sigma)f(x)=\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x(x^\sigma-t^\sigma)^{\alpha-1}t^{\sigma(\eta+1)-1}f(t)dt\ ,\quad \alpha>0\ ,$$

$$(5)\qquad =\frac{x^{1-\sigma(\alpha+\eta+1)}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_a^x(x^\sigma-t^\sigma)^{\alpha}t^{\sigma(\eta+1)-1}f(t)dt\ ,$$

$$-1<\alpha<0\ ,$$

$$(6)\qquad K_{\eta,\alpha}(x,b:\sigma)f(x)=\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\int_x^b(t^\sigma-x^\sigma)^{\alpha-1}t^{\sigma(1-\alpha-\eta)-1}f(t)dt\ ,\quad \alpha>0\ ,$$

$$= -\frac{x^{\sigma(\eta-1)+1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_x^b (t^\sigma - x^\sigma)^\alpha t^{\sigma(1-\alpha-\eta)-1} f(t) dt \;, \qquad -1 < \alpha < 0 \;, \tag{7}$$

where $a < x < b$, $\sigma > 0$.

When $a = 0$, $b = \infty$, these become the extended form of the Erdélyi-Kober operators used in [2] and when $\sigma = 2$ they are the same as the operators defined by Cooke [1].

From the theory of Abel integral equations it follows that the inverse operators are given by

$$I^{-1}_{\eta,\alpha}(a, x\colon \sigma) f(x) = I_{\eta+\alpha,-\alpha}(a, x\colon \sigma) f(x) \;, \tag{8}$$

$$K^{-1}_{\eta,\alpha}(x, b\colon \sigma) f(x) = K_{\eta+\alpha,-\alpha}(x, b\colon \sigma) f(x) \;. \tag{9}$$

We shall also find it convenient to have expressions for integral operators of the type

$$L_{\eta,\alpha}(0, x\colon \sigma) f(x) = I^{-1}_{\eta,\alpha}(a, x\colon \sigma)\, I_{\eta,\alpha}(0, a\colon \sigma) f(x) \;, \qquad 0 < a < x \;, \tag{10}$$

$$M_{\eta,\alpha}(x, b\colon \sigma) f(x) = K^{-1}_{\eta,\alpha}(x, a\colon \sigma)\, K_{\eta,\alpha}(a, b\colon \sigma) f(x) \;, \qquad x < a < b \;. \tag{11}$$

When $0 < \alpha < 1$, we see on using the results (4), (5) and (8) that

$$L_{\eta,\alpha}(0, x\colon \sigma) f(x) = \frac{\sigma x^{1-\sigma(\eta+1)}}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x^\sigma - t^\sigma)^{-\alpha} t^{\sigma-1} dt \int_0^a (t^\sigma - u^\sigma)^{\alpha-1} u^{\sigma(\eta+1)-1} f(u) du \;.$$

Inverting the order of integration and using the result

$$\frac{d}{dx} \int_a^x \frac{t^{\sigma-1} dt}{(x^\sigma - t^\sigma)^\alpha (t^\sigma - u^\sigma)^{1-\alpha}} = \frac{x^{\sigma-1}(a^\sigma - u^\sigma)^\alpha}{(x^\sigma - u^\sigma)(x^\sigma - a^\sigma)^\alpha} \;, \qquad u < a < x,\ 0 < \alpha < 1 \;,$$

we find

$$L_{\eta,\alpha}(0, x\colon \sigma) f(x) = \frac{\sigma \sin(\alpha\pi)}{\pi} \frac{x^{-\sigma\eta}}{(x^\sigma - a^\sigma)^\alpha} \int_0^a \frac{u^{\sigma(\eta+1)-1}(a^\sigma - u^\sigma)^\alpha}{x^\sigma - u^\sigma} f(u) du \;. \tag{12}$$

Similarly we can show that

$$M_{\eta,\alpha}(x, b\colon \sigma) f(x) = \frac{\sigma \sin(\alpha\pi)}{\pi} \frac{x^{\sigma(\alpha+\eta)}}{(a^\sigma - x^\sigma)^\alpha} \int_a^b \frac{u^{\sigma(1-\alpha-\eta)-1}(u^\sigma - a^\sigma)^\alpha}{u^\sigma - x^\sigma} f(u) du \;, \tag{13}$$

where $0 < \alpha < 1$.

When $-1 < \alpha < 0$, the formulae for $L_{\eta,\alpha}$ and $M_{\eta,\alpha}$ are exactly the same as those given by the above equations.

We also have the expressions

$$
\begin{aligned}
(14)\qquad & I_{\eta+\alpha,-\alpha}(0, a\colon \sigma)\, I_{\eta,\alpha}(0, x\colon \sigma) f(x) \\
&= [I^{-1}_{\eta,\alpha}(0, x\colon \sigma) - I^{-1}_{\eta,\alpha}(a, x\colon \sigma)]\, I_{\eta,\alpha}(0, x\colon \sigma) f(x) \\
&= f(x) - I^{-1}_{\eta,\alpha}(a, x\colon \sigma)\, [I_{\eta,\alpha}(0, a\colon \sigma) + I_{\eta,\alpha}(a, x\colon \sigma)] f(x) \\
&= -I^{-1}_{\eta,\alpha}(a, x\colon \sigma) I_{\eta,\alpha}(0, a\colon \sigma) f(x) = -L_{\eta,\alpha}(0, x\colon \sigma) f(x)\ ,
\end{aligned}
$$

$$
(15)\qquad K_{\eta+\alpha,-\alpha}(a, b\colon \sigma)\, K_{\eta,\alpha}(x, b\colon \sigma) f(x) = -M_{\eta,\alpha}(x, b\colon \sigma) f(x)\ .
$$

Two well known results [2] which play an important part in our solution are

$$
(16)\qquad \mathfrak{M}(I_{\eta,\alpha}(0, x\colon \sigma) f(x); s\} = \frac{\Gamma(1 + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)}\, \mathfrak{M}\{f(x); s\}\ ,
$$

$$
(17)\qquad \mathfrak{M}\{K_{\eta,\alpha}(x, \infty\colon \sigma) f(x); s\} = \frac{\Gamma(\eta + s/\sigma)}{\Gamma(\eta + \alpha + s/\sigma)}\, \mathfrak{M}\{f(x); s\}\ .
$$

In what follows we are concerned with three ranges of the variable x, namely

$$
(18)\qquad I_1 = \{x\colon 0 \leqq x < a\},\ I_2 = \{x\colon a < x < b\},\ I_3 = \{x\colon b < x < \infty\}\ ,
$$

and we shall write any function $f(x)$, $x \geqq 0$, in the form

$$
(19)\qquad f(x) = \sum_{i=1}^{3} f_i(x)\ ,
$$

where

$$
(20)\qquad f_i(x) = \begin{cases} f(x),\ x \in I_i\ , \\ 0,\ \text{otherwise}\ , \end{cases} \qquad i = 1, 2, 3\ .
$$

With these definitions it is easily seen that if we evaluate the equations

$$
(21)\qquad g(x) = I_{\eta,\alpha}(0, x\colon \sigma) f(x),\ h(x) = K_{\eta,\alpha}(x, \infty\colon \sigma) f(x)\ ,
$$

on the intervals I_1, I_2 and I_3 respectively, we get

$$
(22)\qquad \begin{aligned}
g_1(x) &= I_{\eta,\alpha}(0, x\colon \sigma) f_1(x)\ , \\
h_1(x) &= K_{\eta,\alpha}(x, a\colon \sigma) f_1(x) + K_{\eta,\alpha}(a, b\colon \sigma) f_2(x) + K_{\eta,\alpha}(b, \infty\colon \sigma) f_3(x)\ ,
\end{aligned}
$$

$$
(23)\qquad \begin{aligned}
g_2(x) &= I_{\eta,\alpha}(0, a\colon \sigma) f_1(x) + I_{\eta,\alpha}(a, x\colon \sigma) f_2(x)\ , \\
h_2(x) &= K_{\eta,\alpha}(x, b\colon \sigma) f_2(x) + K_{\eta,\alpha}(b, \infty\colon \sigma) f_3(x)\ ,
\end{aligned}
$$

$$
(24)\qquad \begin{aligned}
g_3(x) &= I_{\eta,\alpha}(0, a\colon \sigma) f_1(x) + I_{\eta,\alpha}(a, b\colon \sigma) f_2(x) + I_{\eta,\alpha}(b, x\colon \sigma) f_3(x)\ , \\
h_3(x) &= K_{\eta,\alpha}(x, \infty\colon \sigma) f_3(x)\ .
\end{aligned}
$$

3. Solution of the integral equations. Using the notation of equations (19) and (20) we can write the triple integral equations (1) and (2) as

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(\xi + s/\delta)}{\Gamma(\xi + \beta + s/\delta)} \Phi(s); x\right\} = g(x) , \tag{25}$$

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(1 + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)} \Phi(s); x\right\} = f(x) , \tag{26}$$

where $g_1 = g_3 = 0$, f_2 is given and g_2, f_1 and f_3 are unknown functions.

If we write

$$\Phi(s) = \mathfrak{M}\{\phi(x); s\} , \tag{27}$$

and use the formulae (16) and (17) we find that equations (25) and (26) assume the operational form

$$I_{\eta,\alpha}(0, x: \sigma)\phi(x) = f(x) , \tag{28}$$

$$K_{\xi,\beta}(x, \infty: \delta)\phi(x) = g(x) . \tag{29}$$

Using the formulae (8) and (9) and solving the above equations for $\phi(x)$ we obtain

$$\phi(x) = I_{\eta+\alpha,-\alpha}(0, x: \sigma)f(x) \tag{30}$$

$$= K_{\xi+\beta,-\beta}(x, \infty: \delta)g(x) . \tag{31}$$

Now remembering that $g_1 = g_3 = 0$, and using the relations (22), (23) and (24) to evaluate equation (28) on the interval I_1, equation (30) on I_2, equation (31) on I_3, equation (29) on I_2 and equation (31) on I_1, we arrive at the following results

$$f_1(x) = I_{\eta,\alpha}(0, x: \sigma)\phi_1(x) , \tag{32}$$

$$\phi_2(x) = I_{\eta+\alpha,-\alpha}(0, a: \sigma)f_1(x) + I^{-1}_{\eta,\alpha}(a, x: \sigma)f_2(x) , \tag{33}$$

$$\phi_3(x) = K^{-1}_{\xi,\beta}(x, \infty: \delta)g_3(x) = 0 , \tag{34}$$

$$g_2(x) = K_{\xi,\beta}(x, b: \delta)\phi_2(x) , \tag{35}$$

$$\phi_1(x) = K_{\xi+\beta,-\beta}(a, b: \delta)g_2(x) . \tag{36}$$

After eliminating $f_1(x)$ between equations (32) and (33), and eliminating $g_2(x)$ between equations (35) and (36), we find that the functions $\phi_1(x)$ and $\phi_2(x)$ satisfy the pair of simultaneous integral equations

$$\phi_2(x) = -L_{\eta,\alpha}(0, x: \sigma)\phi_1(x) + I^{-1}_{\eta,\alpha}(a, x: \sigma)f_2(x) , \tag{37}$$

$$\phi_1(x) = -M_{\xi,\beta}(x, b:\delta)\phi_2(x) , \tag{38}$$

where we have used the formulae (14) and (15).

From these results it is easily seen that $\phi_2(x)$ can be determined from the Fredholm integral equation of the second kind

$$\phi_2(x) = L_{\eta,\alpha}(0, x: \sigma) M_{\xi,\beta}(x, b: \delta) \phi_2(x) + I_{\eta,\alpha}^{-1}(a, x: \sigma) f_2(x) \; . \tag{39}$$

The solution to the triple integral equations can then be obtained from equations (27), (34), (38) and (39).

As an example we consider the case when $0 < \alpha < 1$, and $-1 < \beta < 0$, or $0 < \beta < 1$; in this instance equation (39) when written out in detail is

$$\begin{aligned} &\phi_2(x) - \int_a^b \phi_2(u) S(x, u) du \\ &\quad = \frac{x^{1-\sigma(\eta+1)}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{t^{\sigma(\alpha+\eta+1)-1}}{(x^\sigma - t^\sigma)^\alpha} f_2(t) dt \; , \end{aligned} \tag{40}$$

where

$$\begin{aligned} S(x, u) = \frac{\sigma\delta}{\pi^2} \sin(\alpha\pi) \sin(\beta\pi) & \frac{x^{-\sigma\eta} u^{\delta(1-\beta-\xi)-1}}{(x^\sigma - a^\sigma)^\alpha (u^\delta - a^\delta)^{-\beta}} \\ & \int_0^a \frac{t^{\sigma(\eta+1)+\delta(\beta+\xi)-1}(a^\sigma - t^\sigma)^\alpha}{(x^\sigma - t^\sigma)(u^\delta - t^\delta)(a^\delta - t^\delta)^\beta} dt \; . \end{aligned} \tag{41}$$

4. An application. Certain mixed boundary value problems [4] may be reduced to the solution of triple integral equations of the type

$$\int_0^\infty \psi(u) J_{2p}(ux) du = 0 \; , \qquad 0 \leqq x < a \; , \quad b < x < \infty \; , \tag{42}$$

$$\int_0^\infty u^{-2n} \psi(u) J_{2q}(ux) du = F(x) \; , \qquad a < x < b \; , \tag{43}$$

where $J_{2p}(ux)$ is the Bessel function of the first kind of order $2p$, $F(x)$ is a prescribed function and $\psi(u)$ is to be determined. When $p = q$ these are the equations investigated by Cooke [1]. We now show, in a fairly straightforward manner, that the above equations can be transformed into equations of the type (1) and (2).

Denoting the Mellin transform of $\psi(u)$ by

$$\mathfrak{M}\{\psi(u); s\} = \Psi(s) \; , \tag{44}$$

and using the result [3]

$$\mathfrak{M}\{\xi^{-2n} J_{2q}(\xi); s\} = 2^{s-1-2n} \frac{\Gamma(q - n + s/2)}{\Gamma(1 + n + q - s/2)} \; , \tag{45}$$

we have, on applying the Faltung theorem for Mellin transforms [3], that the integral equations (42) and (43) can be written in the form

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(p+s/2)}{\Gamma(q-n+s/2)}\Phi(s);\, x\right\} = 0\,, \quad 0 \leqq x < a\,, \quad b < x < \infty\,, \tag{46}$$

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(1+p-s/2)}{\Gamma(1+n+q-s/2)}\Phi(s);\, x\right\} = 2^{1+2n}x^{-2n}F(x)\,, \quad a < x < b\,, \tag{47}$$

where

$$\Phi(s) = 2^{s}\frac{\Gamma(q-n+s/2)}{\Gamma(1+p-s/2)}\Psi(1-s)\,. \tag{48}$$

These are the same as equations (1) and (2) with

$$\begin{aligned} &\sigma = \delta = 2,\ \xi = \eta = p,\ \alpha = q-p+n,\ \beta = q-p-n\,, \\ &f_2(x) = 2^{1+2n}x^{-2n}F(x)\,. \end{aligned} \tag{49}$$

Using the results of the previous section we have therefore that the solution of equations (46) and (47) can be found in terms of a function $\phi(x)$ by

$$\Phi(s) = \mathfrak{M}\{\phi(x);\, s\}\,, \tag{50}$$

where $\phi_3(x) = 0$ and the functions $\phi_1(x)$ and $\phi_2(x)$ are obtained from equations (38) and (39) with the parameters ξ, η, etc. given by equations (49).

Finally, in order to find the solution of the integral equations (42) and (43) in terms of $\phi(x)$, we proceed in the following way.

From equation (44) we have that the solution is

$$\begin{aligned} \psi(u) &= \mathfrak{M}^{-1}\{\Psi(s);\, u\} \\ &= \mathfrak{M}^{-1}\left\{2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)}\mathfrak{M}\{\phi(x);\, 1-s\};\, u\right\}\,, \end{aligned} \tag{51}$$

on using equations (48) and (50). Inverting the order of integration in the last equation we get

$$\begin{aligned} \psi(u) &= \int_0^\infty \phi(x)\mathfrak{M}^{-1}\left\{2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)};\, ux\right\}dx \\ &= \int_0^\infty \left(\frac{ux}{2}\right)^{1+n+p-q}\phi(x)\,J_{p+q-n}(ux)dx\,, \end{aligned} \tag{52}$$

after applying the result (45). When $p = q$ this solution is exactly the same as that found by Cooke [1, pp. 61–62].

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UNIVERSITY OF STRATHCLYDE
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