A NOTE ON ANNIHILATOR IDEALS OF COMPLEX BORDISM CLASSES

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Recent studies of the complex bordism homology theory $\Omega_*^U(\_)$ have shown that for a finite complex $X$ the integer hom-dim $\mathrm{hom}_U \Omega_*^U(X)$ provides a useful numerical invariant measuring certain types of complexity in $X$. Associated to an element $\alpha \in \Omega_*^U(X)$ one has the annihilator ideal $A(\alpha) \subset \Omega_*^U$. Numerous relations between $A(\alpha)$ and hom-dim $\mathrm{hom}_U \Omega_*^U(X)$ are known. In attempting to deal with these invariants it is of course useful to study special cases, and families of special cases. In this note we study the annihilator ideal of the canonical element $\sigma \in \Omega_*^U(X)$ where $X$ is a complex of the form

$$S^{2N} \cup_p e^{2N+1} \cup e^{2N+2n_1-1} \cup \cdots \cup e^{2N+2n_k-1}$$

and $N > n_1, \ldots, n_k > 1$, and $p$ an odd prime. We show that $A(\sigma) \ni [V^{2p^2-2}], \ldots, [V^{2p^2-2}]$, where $[V^{2p^2-2}] \in \Omega_*^{2p^2-2}$ is a Milnor manifold for the prime $p$. This provides another piece of evidence that for such a complex $X$, hom-dim $\mathrm{hom}_U \Omega_*^U(X)$ is 1 or 2.

In [9], [11] and [12] the study of the annihilator ideal of the canonical class $\alpha \in \widetilde{\Omega}_0^U(X)$ in a stable complex $X$ of the form

$$X = S^0 \cup_p e^t \bigcup_{f_i} e^{2n_i-1}$$

played a crucial role in the applications of [9] and [11] to the stable homotopy of spheres. In the closing remarks of [4] it was suggested that more generally for a stable complex of the form (where $p$ is an odd prime)

$$Y = S^0 \cup_p e^t \bigcup_{f_i} e^{2n_i-1} \cdots \bigcup_{f_k} e^{2n_k-1}$$

the annihilator ideal of the canonical class $\alpha \in \widetilde{\Omega}_0^U(Y)$ had the form $(p, [CP(p-1)]^t)$ for some integer $t$, and that hom-dim $\mathrm{hom}_U \Omega_*^U(Y) \leq 2$. Our objective in this note is to make the following elementary contribution to these matters.

**Theorem.** Let $Y$ be a stable complex of the form

$$S^0 \cup_p e^t \bigcup_{f_1} e^{2n_1-1} \bigcup_{f_2} e^{2n_2-1} \cdots \bigcup_{f_k} e^{2n_k-1}$$

where $p$ is odd prime. Let $\sigma \in \widetilde{\Omega}_0^U(Y)$ denote the canonical class. Then
for any $s > 1$, where $A(\sigma) \subseteq \Omega^u_*$ denotes the annihilator ideal of $\sigma$, and $[V^{2p^s-2}] \in \Omega^{u-2}_{2p^s-2}$ is a Milnor manifold for the prime $p$.

Remark. An easy computation using Landweber-Novikov operations [1] [2] [5] [7] shows, for any stably spherical bordism element $\sigma \in \Omega^u_*(X)$ of additive order $p$, on a complex $X$, that

$$A(\sigma) \subseteq (p, [V^{2p-2}], (V^{2p^2-2}], \cdots).$$

It is therefore not unreasonable to ask if some particular Milnor manifold $[V^{2p^s-2}] \in A(\sigma)$.

The proof of this result follows closely the developments of [3] [4] [10]. The fundamental fact that we shall use is implicit in both [3] and [10] and may be stated as follows:

**Proposition.** Let $X$ be a finite complex and suppose that $\sigma \in \Omega^u_*(X)$ is a stably spherical class and satisfies the following conditions

1. $p\sigma = 0$, $p$ a prime
2. $\mu_p(\sigma) \neq 0 \in \tilde{H}_*(X; \mathbb{Z}_p)$ where $\mu_p$ is the composite $\mu_p : \tilde{\Omega}^u_*(X) \rightarrow \tilde{H}_*(X; \mathbb{Z}) \rightarrow \tilde{H}_*(X; \mathbb{Z}_p)$ of the Thom homomorphism $\mu$ and reduction mod $p$.

If $[V^{2p^s-2}] \in A(\sigma)$ then

$$S_i\beta \in \tilde{H}^{n+2p^s-1}(X; \mathbb{Z}_p)$$

where

- $n = \deg \sigma$
- $u \in \tilde{H}^n(X; \mathbb{Z}_p)$ is dual to $\mu_p(\sigma)$ ($\mathfrak{A}(p)$ the mod $p$ Steenrod algebra).
- $\beta \in \mathfrak{A}(p)$ the Bockstein
- $S_i \in \mathfrak{A}(p)$ the primitive element of degree $2p^i-2$.

The proof of the preceding proposition may be deduced directly from [10; 1.2] as in §2 of [10] or [3; 2.4].

**Proof of Theorem.** Suppose to the contrary that $[V^{2p^s-2}] \in A(\sigma)$ for some $s > 1$.

Let us denote by $s_\alpha, \alpha = (a_i, \cdots)$ the Landweber-Novikov operation corresponding to the sequence $\alpha$. (See [1] [2] [5] [7] or [11] for information about Landweber-Novikov operations). The following formula for the action of the Landweber-Novikov operations on $\Omega^u_*$ may be found in [11; 2.1] (and the succeeding remark):

$$s_{p^s-p^{s-1}-p^{s-2}}[V^{2p^s-2}] \equiv [V^{2p^{s-1}-2}] \mod p.$$
Applying the Cartan formula to the fact that $\sigma$ is a spherical class we obtain:

$$[V^{2p^s-1-2}] \sigma = (s_{p,p^{s-1-p^{s-2}}}[V^{2p^s-2}] \sigma) = s_{p,p^{s-1-p^{s-2}}}([V^{2p^s-2}] \sigma) = s_{p,p^{s-1-p^{s-2}}} (0) = 0 .$$

Hence $[V^{2p^s-1-2}] \in A(\sigma)$. In this way we see that (recall we may choose $[V^{2p^s-2}] = [CP(p-1)]$).

$p, [V^{2p^s-2}], \ldots, [V^{2p^s-2}] \in A(\sigma)$.

Note next that

$$\tilde{H}^i(Y; Z_p) \cong \{Z: \ i = 0, 1, 2n_1-1, \ldots, 2n_k-1 \} \cup \{0: \text{otherwise.} \}$$

For $i = 0, 1, 2n_1-1, \ldots, 2n_k-1$, let $e_i \in \tilde{H}^i(Y; Z_p)$ denote a nonzero class. Note that we may choose $e_i = \beta e_0$.

Applying the preceding proposition to the fact that $[CP(p-1)] \in A(\sigma)$ we conclude that (recall $S_1 = P^i$)

$$P^i \beta e_0 \neq 0 \in \tilde{H}^{2p-1}(Y; Z_p) .$$

Hence we may assume that $n_i = p$ and

$$e_{2p-1} = e_{2n_i-1} = P^i \beta e_0 .$$

Next apply the preceding proposition to the fact that $[V^{2p^2-2}] \in A(\sigma)$ to conclude that

$$S_2 \beta e_0 \neq 0 \in H^{2p^2-1}(Y; Z_p) .$$

Recall that

$$S_2 = P^pP^i - P^iP^p .$$

Thus

$$P^p \beta e_0 - P^i \beta e_0 \neq 0 \in \tilde{H}^{2p^2-1}(Y; Z_p) .$$

Recall now the following Adem relations:

$$P^i P^p = -P^{p+1}$$

$$P^i \beta P^p = P^{p+1} \beta = P^i P^p \beta .$$

Therefore
But

\[ P^1 P^p \beta e_0 = P^1 \beta P^p e_0 . \]

and hence

\[ P^1 P^p \beta e_0 = 0 . \]

Therefore we conclude that

\[ P^p P^1 \beta e_0 \neq 0 \in \tilde{H}^{2p(p-1)}(Y; Z_p) . \]

Next we propose to apply the factorization theorem of [6], [8] to deduce a contradiction. Consider therefore

\[ P^1 P^1 \beta e_0 . \]

Recall the Adem relations

\[
\begin{align*}
P^1 P^1 &= 2P^2 \\
P^2 \beta - \beta P^2 &= \beta P^1 P^1 - P^1 \beta P^1 .
\end{align*}
\]

We have therefore

\[ P^1 P^1 \beta e_0 = 2P^2 \beta e_0 = 2(\beta P^2 e_0 + \beta P^1 e_0 - P^1 \beta P^1 e_0) . \]

Now note that

\[
\begin{align*}
P^2 e_0 &= 0 \\
P^1 e_0 &= 0
\end{align*}
\]

for dimensional reasons. Therefore

\[ P^1 P^1 \beta e_0 = 2(0) = 0 . \]

The class \( P^1 \beta e_0 \in \tilde{H}^{2p-1}(Y; Z_p) \) therefore satisfies

\[
\begin{align*}
P^1(P^1 \beta e_0) &= 0 \\
\beta(P^1 \beta e_0) &= 0
\end{align*}
\]

(the latter from dimensional considerations). We may therefore apply the formula of [6] [8] to conclude that

\[ P^p P^1 \beta e_0 = \beta \Lambda(P^1 \beta e_0) + 2P^{p-2} \mathcal{R}(P^1 \beta e_0) \]

modulo a suitable indeterminacy, where \( \Lambda \) and \( \mathcal{R} \) are secondary operations of degree \( 2p(p-1)-1 \) and \( 4(p-1) \) respectively. Note that

\[ \Lambda P^1 \beta e_0 \in \tilde{H}^{2p^2-2}(Y; Z_p) = 0 \]

and hence

\[ P^p P^1 \beta e_0 = 2P^{p-2} \mathcal{R} P^2 \beta e_0 . \]
modulo a suitable indeterminacy. And so it remains for us to eliminate the possibility that
\[ P^{p-2} \not\equiv P^2 \beta e_0 \neq 0. \]

This will require a careful analysis of the cell structure of \( Y \) thru dimensions \( 6p - 6 \). Recall that
\[ Y = S^0 \bigcup_{p} e^i \cdots. \]

Let \( M = S^0 \bigcup_{p} e^i \). It follows from [14] that (N. B. it is necessary to divide into two cases, according to \( p > 3 \) or \( p = 3 \))
\[ \pi^*_i(M) = \begin{cases} Z_p: & i = 0, 2p - 3, 2p - 2, 4p - 5, 4p - 4, 6p - 7 \text{ or } 6p - 6 \\ 0: & \text{otherwise for } 0 \leq i \leq 6p - 6. \end{cases} \]

A stable map
\[ \alpha: S^{2p-2} \longrightarrow M \]
represents a nonzero element of \( \pi_{2p-3}^*(M) \) iff
\[ P^i \beta: \tilde{H}^i(M \bigcup_{a} e^{2p-1}; Z_p) \longrightarrow \tilde{H}^{2p-1}(M \bigcup_{a} e^{2p-1}; Z_p) \]
is an isomorphism. Thus from the preceding analysis of the fact that \([CP(p - 1)] = [V^{2p-2}] \in A(\sigma)\) we find
\[ Y = S^0 \bigcup_{p} e^i \bigcup_a e^{2p-1} \cdots \]
where \([\alpha] \in \pi_{2p-3}^*(S^0 \bigcup_{p} e^i)\) is a generator. Next we form the cofibration
\[ S^{2p-2} \xrightarrow{\alpha} M \xrightarrow{q} N \]
defining \( N \) as \( S^0 \bigcup_{p} e^i \bigcup_a e^{2p-1} \). Note that
\[ \pi^*_{2p-2}(N) = 0 \]
and therefore the \( 2p-2 \) skeleton of \( Y \) has the form
\[ N \vee (S^{2p-1} \vee \cdots \vee S^{2p-1}). \]

We propose now to calculate the homotopy of \( N \) in low dimensions. From the definition of \( N \) we obtain an exact sequence
\[ \cdots \pi^*_i(S^{2p-2}) \xrightarrow{\alpha_*} \pi^*_i(M) \xrightarrow{q_*} \pi^*_i(N) \xrightarrow{\partial_*} \pi^*_{i-2}(S^{2p-2}) \cdots. \]
Consulting [14] and the little table of the homotopy of \( M \) above we see that only a few values of \( i \) lead to \( p \)-primary information. The first such section is:
\[ \cdots \xrightarrow{\alpha_*} \pi^*_i(S^{2p-2}) \xrightarrow{q_*} \pi^*_i(M) \xrightarrow{q_*} \pi^*_i(N) \xrightarrow{\partial_*} \pi^*_i(S^{2p-2}) \cdots. \]
Now note that
\[ \alpha_*: \pi_{i_{p-5}}^s(S^{2p-2}) \to \pi_{i_{p-5}}^s(M) \]
is an isomorphism on the \( p \)-components which are \( \mathbb{Z}_p \). Thus our
sequence divides into
\[
0 \to \pi_{4p-4}^s(M) \longrightarrow \pi_{4p-4}^s(N) \longrightarrow 0 \\
0 \to \pi_{4p-5}^s(S^{2p-2}) \longrightarrow \pi_{4p-5}^s(M) \longrightarrow 0 \\
0 \to \pi_{4p-6}^s(N) \longrightarrow \pi_{4p-6}^s(S^{2p-2}) = 0 .
\]
We thus find that
\[ \pi_i^s(N) \cong \begin{cases} 
\mathbb{Z}_p: & i = 4p-4 \\
\text{a group of order prime to } p: & 2p-1 < i < 4p-4 .
\end{cases} \]
It will be important to remember that the generator of \( \pi_{4p-4}^s(N) \) comes
from \( \pi_{4p-4}^s(M) \). Denote this generator by \( \beta \). In a similar manner we
find that
\[ \pi_i^s(N) \cong \begin{cases} 
\mathbb{Z}_p: & i = 6p-6 \\
\text{a group of order prime to } p: & 4p-4 \leq i \leq 6p-6 ,
\end{cases} \]
and the generator of \( \pi_{6p-6}^s(N) \) comes from \( \pi_{6p-6}^s(M) \). Denote this gen-
erator by \( \gamma \). Observe that since \( \beta \) and \( \gamma \) pull off the top cell of \( N \) that
\[ P^i: H^{2p-1}(N \sqcup \mathbb{Z}_p; e^{4p-3}; Z_p) \longrightarrow H^{4p-3}(N \sqcup \mathbb{Z}_p; e^{4p-3}; Z_p) \]
and
\[ \mathcal{B}: H^{2p-1}(N \cup \mathbb{Z}_p; e^{6p-3}; Z_p) \longrightarrow H^{6p-3}(N \cup \mathbb{Z}_p; e^{6p-3}; Z_p) \]
are zero. The first of these shows that
\[ \mathcal{B}: H^{2p-1}(e^{6p-5} \cup \mathbb{Z}_p; e^{4p-3}; Z_p) \longrightarrow H^{6p-3}(e^{6p-5} \cup \mathbb{Z}_p; e^{4p-3}; Z_p) \]
is again defined but is still zero, modulo zero. (N.B. The indeter-
mminacy of the operation \( \mathcal{B} \) in the sum of the images of the operations
\( 2P^1 \beta - \beta P^1 \) and \( P^2 \), which vanish for \( e^{6p-5} \cup \mathbb{Z}_p; e^{4p-3} \) for di-
minimal reasons).

Let us denote by
\[ r: N \vee (S^{2p-1} \vee \cdots \vee S^{2p-1}) \longrightarrow M \]
the natural map. The analysis preceding shows that thru the \( 6p-6 \)
skeleton \( Y^{6p-6} \), of \( Y \), there is a map
inducing an isomorphism of $\mathbb{Z}_p$ cohomology thru dimensions 0 and 1 and an injections thru dimensions $6p-6$. The particular space on the right depending on the cells of $Y$ occurring.

Therefore $R P^1 \beta e_0 \neq 0 \in H^{6p-5}(Y; \mathbb{Z}_p)$ iff the corresponding result is true for one of the spaces $N, N \cup \beta e^{6p-3}, N \cup \beta e^{6p-5}, e^{6p-5} \cup \beta e^{6p-3}$. However, in constructing these spaces we carefully checked that $R P^1 \beta e_0 = 0$ modulo 0. Therefore

$$R P^1 \beta e_0 = 0 \in H^{6p-5}(Y; \mathbb{Z}_p)$$

modulo 0 and hence

$$P^p P^1 \beta e_0 = P^{p-2} R P^1 \beta e_0 = 0 \in H^{2p^2-1}(Y; \mathbb{Z}_p)$$

modulo 0 from which we conclude $[V^{2p^2-2}] \in A(\sigma)$. But as we saw at the beginning of the proof this is contrary to the assumption that $[V^{2p^2-2}] \in A(\sigma)$ for some $s > 1$ and the theorem is established. **

**Question.** Does there exist a simple Adem type relation that would show $R P^1 \beta e_0 = 0$ modulo 0 in $H^{6p-5}(Y; \mathbb{Z}_p)$ without going through the cell by cell analysis above?

**Corollary.** Let $Y$ be a stable complex of the form

$$Y = S^\infty \cup \beta e^1 \cup f_1 e^{2^2-1} \cdots \cup f_k e^{2^k-1}$$

where $p$ is an odd prime. Let $\sigma \in \Omega^\ell_0 (Y)$ denote the canonical class. Then for each integer $s > 1$ the class $[V^{2p^s-2}] \sigma$ is not stably spherical, that is, for $s > 1$

$$[V^{2p^s-2}] \sigma \notin \text{Im } \{\Omega_*^\infty (Y) \longrightarrow \Omega_*^\infty (Y)\}.$$ **

**References**


Received February 5, 1971.

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Richard Davis Anderson and Thomas Ashland Chapman, *Extending homeomorphisms to Hilbert cube manifolds* ........................................... 281
Nguyen Huu Anh, *Restriction of the principal series of $\text{SL}(n, \mathbb{C})$ to some reductive subgroups* .................................................. 295
David W. Boyd, *Indices for the Orlicz spaces* .............................. 315
William Garfield Bridges, *The polynomial of a non-regular digraph* .... 325
H. A. Çelik, *Commutative associative rings and anti-flexible rings* ...... 351
Hsin Chu, *On the structure of almost periodic transformation groups* .... 359
David Allyn Drake, *The translation groups of n-uniform translation Hjelmslev planes* ................................................................. 365
Michael Benton Freeman, *The polynomial hull of a thin two-manifold* ... 377
Anthony Alfred Gioia and Donald Goldsmith, *Convolutions of arithmetic functions over cohesive basic sequences* .......................... 391
Leslie C. Glaser, *A proof of the most general polyhedral Schoenflies conjecture possible* ............................................................... 401
Thomas Lee Hayden and Ted Joe Suffridge, *Biholomorphic maps in Hilbert space have a fixed point* ...................................................... 419
Roger Alan Horn, *Schlicht mappings and infinitely divisible kernels* ...... 423
Norman Ray Howes, *On completeness* ........................................ 431
Hideo Imai, *Sario potentials on Riemannian spaces* .......................... 441
A. A. Iskander, *Subalgebra systems of powers of partial universal algebras* ....................................................................................... 457
Barry E. Johnson, *Norms of derivations of $L^1(X)$* .......................... 465
David Clifford Kay and Eugene W. Womble, *Axiomatic convexity theory and relationships between the Carathéodory, Helly, and Radon numbers* ................................................................. 471
Constantine G. Lascarides, *A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer* .......................... 487
C. N. Linden, *On Blaschke products of restricted growth* ................. 501
John S. Lowndes, *Some triple integral equations* ............................. 515
Declan McCartan, *Bicontinuous preordered topological spaces* ............ 523
S. Moedomo and J. Jerry Uhl, Jr., *Radon-Nikodým theorems for the Bochner and Pettis integrals* ......................................................... 531
Calvin Cooper Moore and Joseph Albert Wolf, *Totally real representations and real function spaces* ............................................... 537
Reese Trego Prosser, *A form of the moment problem for Lie groups* .... 543
Larry Smith, *A note on annihilator ideals of complex bordism classes* .... 551
Joseph Warren, *Meromorphic annular functions* ................................ 559