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MEROMORPHIC ANNULAR FUNCTIONS

JOSEPH WARREN

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The purpose of this paper is to present a definition of meromorphic annular functions which includes the definition of holomorphic annular functions. Several equivalent conditions for meromorphic annular functions are given.

2. Preliminary definitions and remarks. Let D be the disk $|z| < 1$ and C the circle $|z| = 1$. We shall, henceforth, assume that the function $f(z)$ is meromorphic in D .

A boundary path in D is the image of the unit interval $0 \leq t < 1$ under a continuous function $z = z(t)$ from $0 \leq t < 1$ into D such that $\lim_{t \rightarrow 1} |z(t)| = 1$. A spiral in D is a boundary path with the additional condition that $\lim_{t \rightarrow 1} \arg z(t) = +\infty$ or $-\infty$ for any branch of the argument of $z(t)$.

The set $L(\lambda) = \{z \mid |f(z)| = \lambda, 0 < \lambda < \infty\}$ is called a level set for the function f and a component of $L(\lambda)$ is called a level curve. It is known [6, Prop. 1] that if $C(\lambda)$ is a level curve which does not contain any zeros of $f'(z)$, then $C(\lambda)$ is either a Jordan curve contained in D or the union of two disjoint boundary paths. If $\lambda = 0$ or $\lambda = \infty$, then $L(\lambda)$ corresponds to the set of zeros or poles, respectively, of $f(z)$.

The function $f(z)$ has the asymptotic value a (allowing $a = \infty$) if there exists a boundary path $z = z(t)$, $0 \leq t < 1$, such that $\lim_{t \rightarrow 1} f(z(t)) = a$.

The following definition will be taken for the definition of meromorphic annular functions.

DEFINITION 1. Let $f(z)$ be a nonconstant meromorphic function in D and let $\{J_n\}$ be a sequence of Jordan curves with J_1 contained in the interior of J_n for $n = 2, 3, 4, \dots$ such that either

$$\lim_{n \rightarrow \infty} \max_{z \in J_n} |f(z) - a| = 0,$$

for a finite value a , or, if $a = \infty$,

$$\lim_{n \rightarrow \infty} \min_{z \in J_n} |f(z)| = \infty.$$

If, in addition, f has an asymptotic value, then f will be called an annular function with respect to a .

The class of annular functions with respect to a will be denoted by $\mathcal{A}(a)$.

REMARK 1. It is proved in Theorem 1 of [7] that if $f \in \mathcal{A}(a)$, then given any r , $0 < r < 1$, there exists an integer N such that if $n \geq N$, then the disk $|z| \leq r$ is contained in the interior of J_n . In such a case the sequence $\{J_n\}$ is said to converge uniformly to the boundary C .

REMARK 2. A subsequence $\{J_{n_k}\}$ can be selected such that if $k \neq j$ then $J_{n_k} \cap J_{n_j} = \phi$.

From these two remarks it may be assumed that the members of the sequence $\{J_n\}$ of Definition 1 are pairwise disjoint and that the sequence tends uniformly to C .

REMARK 3. It is evident that if $f \in \mathcal{A}(a)$ then the asymptotic value assumed to exist in Definition 1 is a .

REMARK 4. If $a \neq b$, then $\mathcal{A}(a) \cap \mathcal{A}(b) = \phi$. The function f is an $\mathcal{A}(a)$ if and only if $1/f \in \mathcal{A}(1/a)$.

REMARK 5. If f is holomorphic and annular in the old sense [1, 340] then there exists a sequence of Jordan curves $\{J_n\}$ which tend to the circle C and on which f tends uniformly to ∞ . Since every holomorphic function has an asymptotic value, which in this case must be ∞ , it is seen that $f \in \mathcal{A}(\infty)$. Thus there exists a function in $\mathcal{A}(0)$; the reciprocal of any function annular in the old sense.

The following definitions are needed.

DEFINITION 2. If the nonconstant meromorphic function f in D has the asymptotic value a on a spiral asymptotic path, then f is a spiral function with respect to a .

The class of spiral function with respect to a will be denoted by $\mathcal{S}(a)$.

DEFINITION 3. If the nonconstant and meromorphic function f in D is bounded away from a on a spiral boundary path, then f is said to be in the Valiron class with respect to a , provided f has the asymptotic value a .

The class of such functions will be denoted by $\mathcal{V}(a)$.

REMARK 6. $\mathcal{V}(a) \subset \mathcal{S}(a)$.

DEFINITION 4. The function $f(z)$ is in the class $\mathcal{L}'(a)$ if f is nonconstant and meromorphic in D and has the asymptotic value a as well as the following property: In the case of a finite value a ,

every level curve of $f(z) - a$ which is disjoint from the zeros of $f'(z)$ is a compact set in D , or, in the case of $a = \infty$, every level curve of $f(z)$ which is disjoint from the zeros of $f'(z)$ is a compact set in D .

DEFINITION 5. Let $\mathcal{L}(a)$ be the class of functions f such that f is in $\mathcal{S}'(a)$ and such that every level curve of $f(z) - a$ (or $f(z)$ if $a = \infty$) is a compact set in D .

It will be shown that $\mathcal{L}(a) = \mathcal{L}'(a)$.

DEFINITION 6. A tract $\{D(\varepsilon), a\}$ for the meromorphic function f in D associated with the value a is a set of non-void domains $D(\varepsilon)$ each of which is a component of $\{z \mid |f(z) - a| < \varepsilon\}$, or $\{z \mid |f(z)| > 1/\varepsilon\}$ if $a = \infty$, such that $D(\varepsilon) \subset D(\varepsilon')$ if $\varepsilon < \varepsilon'$ and $\bigcap_{\varepsilon < 0} D(\varepsilon) = \phi$.

3. Equivalences for $\mathcal{A}(a)$. The following theorem gives the main equivalences for the class $\mathcal{A}(a)$ and corresponds to Theorems 1 and 3 of [6].

THEOREM 1. $\mathcal{A}(a) = \mathcal{S}(a) - \mathcal{V}(a) = \mathcal{L}'(a) = \mathcal{L}(a)$.

Proof. To prove the theorem in the most economical way we prove the chain of containments $\mathcal{A}(a) \subset \mathcal{S}(a) - \mathcal{V}(a) \subset \mathcal{L}'(a) \subset \mathcal{A}(a) \subset \mathcal{L}(a) \subset \mathcal{L}'(a)$.

First, let $f \in \mathcal{A}(a)$, let T be the asymptotic path on which f tends to a (see Remark 3), and let $\{J_n\}$ be the sequence of Jordan curves of Definition 1 on which f tends uniformly to a . Using the same construction as in Theorem 2 of [6] a spiral may be constructed on which f has the asymptotic value a . Thus $f \in \mathcal{S}(a)$. Evidently every boundary path intersects members of $\{J_n\}$ for all sufficiently large n so that f cannot be bounded away from a on any spiral. Since f has the asymptotic value a , f is not in $\mathcal{V}(a)$ and is in $\mathcal{S}(a) - \mathcal{V}(a)$.

Now let $f \in \mathcal{S}(a)$ and let $C(\lambda)$ be a level curve of $f(z) - a$ which contains no zeros of $f'(z)$. If $C(\lambda)$ is not a Jordan curve in D , then it consists of two boundary paths (spirals) on which f is bounded away from a , and we may conclude that $f \in \mathcal{V}(a)$. Therefore, if $f \in \mathcal{S}(a) - \mathcal{V}(a)$, then each level curve $C(\lambda)$ of $f(z) - a$ containing no zeros of $f'(z)$ must be a Jordan curve in D , and hence $f \in \mathcal{L}'(a)$ and we obtain $\mathcal{S}(a) - \mathcal{V}(a) \subset \mathcal{L}'(a)$.

Let $f \in \mathcal{L}'(a)$. Because f has the asymptotic value a , there is a tract $\{D(\varepsilon), a\}$ associated with a . Choose a sequence $\{\varepsilon_n\}$ of positive numbers such that $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$ and the level set $\{z \mid |f(z) - a| = \varepsilon_n\}$, or $\{z \mid |f(z)| = 1/\varepsilon_n\}$ if $a = \infty$, does not contain any zeros of $f'(z)$

for $n = 1, 2, \dots$.

For each n , $D(\varepsilon_n)$ contains an asymptotic path with asymptotic value a so that it cannot be contained within a Jordan curve in D . Thus the set $D(\varepsilon_n)$ is $|z| < 1$ with a countable or finite number of Jordan domains removed. Otherwise the boundary of $D(\varepsilon_n)$ would contain a level curve which is not a compact set in D , contrary to the hypothesis that $f \in \mathcal{L}'(a)$.

If $D(\varepsilon_n) = D$ for every n then $|f(z) - a| < \varepsilon_n$ (or $|f(z)| > 1/\varepsilon_n$) in D for every n and $f(z)$ is identically constant contrary to the definition of $\mathcal{L}'(a)$. Thus there exists an integer n_1 and a point $z_1 \in D$ which is not in $D(\varepsilon_{n_1})$. Let J_1 be the Jordan curve in D which contains z_1 in its interior and which is a component of the boundary of $D(\varepsilon_{n_1})$. Let J_2 be the Jordan curve which contains z_1 in its interior and is a boundary component of $D(\varepsilon_{n_1+1})$. Because of the definition of tract, $D(\varepsilon_{n_1+1}) \subset D(\varepsilon_{n_1})$ which implies that J_2 contains J_1 in its interior. Continuing in the same manner we obtain a sequence of Jordan curves $\{J_n\}$ such that J_n contains J_1 in its interior for $n = 2, 3, 4, \dots$ and such that $|f(z) - a| = \varepsilon_n$ (or $|f(z)| = 1/\varepsilon_n$) for all $z \in J_n$, $n = 1, 2, \dots$. Since f is not a constant, J_n tends uniformly to C , or $\min_{z \in J_n} |z| \rightarrow 1$ as $n \rightarrow \infty$, because of [7, Theorem 1]. Thus f has an asymptotic value and has the sequence $\{J_n\}$ with all the properties of Definition 1 so that f is in $\mathcal{A}(a)$.

If $f \in \mathcal{A}(a)$, then it is easy to see that any level curve of $f(z) - a$ is contained inside one of the Jordan curves J_n of Definition 1 and is thus compact in D . Thus $f \in \mathcal{L}'(a)$.

Finally, $\mathcal{L}(a) \subset \mathcal{L}'(a)$ by definition, and the proof of Theorem 1 is complete. There is one other characterization of the set $\mathcal{A}(a)$ which was suggested to me by J. Choike.

COROLLARY. *The function f is in $\mathcal{A}(a)$ if and only if f has an asymptotic value and every boundary path contains a sequence of points a_n such that $\lim_{n \rightarrow \infty} f(z_n) = a$ and $\lim_{n \rightarrow \infty} |z_n| = 1$*

Proof. If $f \in \mathcal{A}(a)$ the conclusion follows immediately.

Let $f \notin \mathcal{A}(a)$. Then by Theorem 1 there exists a level curve $C(\lambda)$ of the function $f(z) - a$, or $f(z)$ if $a = \infty$, which is disjoint from the zeros of $f'(z)$ and is not a Jordan curve in D . Hence, $C(\lambda)$ contains a boundary path T on which $|f(z) - a| = \lambda$, or $|f(z)| = \lambda$ if $a = \infty$, and so there does not exist a sequence $z_n \in T$ such that $\lim_{n \rightarrow \infty} f(z_n) = a$ and f fails to satisfy the conditions of the corollary. This completes the proof.

4. **A short proof of a corollary of McMillan.** The proof given in this section is elementary in the sense that it uses only the classical results of Fatou and F. and M. Riesz. The theorem of McMillan [4, p. 151] to which this corollary refers is very complicated and uses many measure theoretic concepts.

The method of proof uses a result of MacLane [3, p. 13] which was used to prove several results in [7].

DEFINITION 7. The end of the tract $\{D(\varepsilon), a\}$ is $\bigcap_{\varepsilon>0} \bar{D}(\varepsilon)$ where $\bar{D}(\varepsilon)$ represents the closure of $D(\varepsilon)$.

THEOREM 2 (McMillan). *If $f(z)$ is a holomorphic function in D which has a finite number of tracts, the union K of the ends of the tracts associated with ∞ is the circle $C: |z| = 1$.*

Proof. Let $T_1(\infty) = \{D_1(\varepsilon), \infty\}$, $T_2(\infty) = \{D_2(\varepsilon), \infty\}$, \dots , $T_m(\infty) = \{D_m(\varepsilon), \infty\}$, $T_1(a_1)$, $T_2(a_2)$, \dots , $T_p(a_p)$, be the tracts for f where $a_i \neq \infty$, $i = 1, 2, \dots, p$.

Assume the contrary of the conclusion: that is $K \neq C$. Because K is closed there exists a disk N about a point of C such that $K \cap \bar{N} = \phi$. If for some i and every $\varepsilon > 0$, $\bar{D}_i(\varepsilon) \cap \bar{N} \neq \phi$, where $D_i(\varepsilon)$ is the set of domains for $T_i(\infty)$, select a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ and a sequence $\{z_n\}$ such that $z_n \in \bar{D}_i(\varepsilon_n) \cap \bar{N}$. By Definition 6 $\{z_n\}$ has a limit point $\zeta \in C$ which is also in \bar{N} . Then $\zeta \in \bigcap_{j=1}^{\infty} \bar{D}_i(\varepsilon_j) \subset K$, in violation of $K \cap \bar{N} = \phi$. Thus for each $n = 1, 2, \dots, m$ there exists an $\varepsilon_n > 0$ such that $\bar{D}_n(\varepsilon_n) \cap \bar{N} = \phi$. For $\varepsilon = \text{Min}\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$, we have $\bar{D}_n(\varepsilon) \cap \bar{N} = \phi$, for each n , $n = 1, 2, \dots, m$.

If $f(z)$ were bounded in $N \cap D$, then $f(z)$ has radial limit almost everywhere on $N \cap C$ by the theorem of Fatou [2]. These limits must be selected from $\{a_1, a_2, \dots, a_p\}$. Let A_i be the set of $\zeta \in N \cap C$ for which f has radial limit a_i . By the F. and M. Riesz theorem [5] the measure of A_i is 0. Hence the measure of $\bigcup_{i=1}^p A_i$ is also 0 and $f(z)$ has radial limit on at most a set of measure 0 on $N \cap C$. Thus $f(z)$ is unbounded in $N \cap D$ and it is possible to choose $z_0 \in N \cap D$ such that $|f(z_0)| > 1/\varepsilon$, $f'(z_0) \neq 0$, and $f(z_0) \neq a_i$ for $i = 1, 2, \dots, p$.

By methods of MacLane [3, p. 13] there exists an arc T from $f(z_0)$ on the Riemann surface of f^{-1} which ends at ∞ . The arc T can be chosen so that its projection in the w -plane is a ray on which $|w| \geq |f(z_0)|$. The inverse image of T has a component γ which contains z_0 . Because $|f(z)| \geq |f(z_0)| \geq 1/\varepsilon$ for all $z \in \gamma$ and because γ is an asymptotic path with ∞ as asymptotic value, $\gamma \subset D_n(\varepsilon)$ for some n between 1 and m . This implies $z_0 \in D_n(\varepsilon) \cap N$. But it has been established that $D_n(\varepsilon) \cap N = \phi$. This contradiction implies that the assumption $K \neq C$ is false and the theorem is proved.

REMARK. The proof just given goes through for holomorphic functions with finite tracts associated with ∞ and a countable number of tracts associated with finite values. By another corollary of McMillan [4, p. 151] no such function exists. If $f(z)$ has finite tracts associated with ∞ and infinite tracts, then f has point asymptotic values (values which are approached along a path that ends at a point of C) on a set of positive measure.

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