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UNIQUELY REPRESENTABLE SEMIGROUPS ON THE TWO-CELL

J. T. BORREGO, HASKELL COHEN AND ESMOND ERNEST DEVUN

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A semigroup S is said to be uniquely representable in terms of two subsets X and Y of S if $X \cdot Y = Y \cdot X = S$, $x_1 y_1 = x_2 y_2$ is a nonzero element of S implies $x_1 = x_2$ and $y_1 = y_2$, and $y_1 x_1 = y_2 x_2$ is a nonzero element of S implies $y_1 = y_2$ and $x_1 = x_2$ for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. A semigroup S is said to be uniquely divisible if for each $s \in S$ and every positive integer n there exists a unique $z \in S$ such that $z^n = s$. Theorem. If S is a uniquely divisible semigroup on the two-cell with the set of idempotents of S being a zero for S and an identity for S , then S is uniquely representable in terms of X and Y where X and Y are isomorphic copies of the usual unit interval and the boundary of S equals X union Y . Corollary. If S is a uniquely divisible semigroup on the two-cell and if S has only two idempotents, a zero and an identity, then the nonzero elements of S form a cancellative semigroup.

A semigroup S is said to be uniquely representable in terms of two subsets X and Y of S if $X \cdot Y = Y \cdot X = S$, $x_1 y_1 = x_2 y_2$ is a nonzero element of S implies $x_1 = x_2$ and $y_1 = y_2$, and $y_1 x_1 = y_2 x_2$ is a nonzero element of S implies $y_1 = y_2$ and $x_1 = x_2$ for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. A semigroup S is said to be uniquely divisible if for every $s \in S$ and every positive integer n there exists a unique $z \in S$ such that $z^n = s$.

The primary purpose of this paper is to show that if S is a uniquely divisible semigroup on two-cell with the set of idempotents of S being a zero for S and an identity for S , then S is uniquely representable in terms of X and Y where X and Y are isomorphic copies of the usual unit interval and the boundary of S equals X union Y . As a corollary to this theorem we shall prove a conjecture of D. R. Brown, that if S is a uniquely divisible semigroup on the two-cell and if S has only two idempotents, a zero and an identity, then the nonzero elements of S form a cancellative subsemigroup of S .

NOTATION. Throughout S will be a uniquely divisible semigroup on the two-cell with $E(S)$ (the set of idempotents of S) = $\{0, 1\}$ where 0 is the zero for S and 1 is the identity for S . It is well known that the boundary of S is the union of two usual threads X and Y with $X \cap Y = \{0, 1\}$ and $S = X \cdot Y = Y \cdot X$. Intervals containing x will represent segments of X and intervals with y shall stand for segments of Y . For a positive integer n , $s^{1/n}$ will denote the unique n th root of s in S .

The authors would like to thank the referee for pointing out the following result due to J. D. Lawson and M. Friedberg and which appears in [2].

LEMMA 1. *If T is a uniquely divisible semigroup with $E(T) = \{0, 1\}$, then T has no zero divisors.*

Proof. Suppose $ab = 0$ for some $a, b \in T$, $a \neq 0$. Then $(ba)^2 = b(ab)a = 0$, hence $ba = 0$. Thus $0 = ab = a^{1/2}(a^{1/2}b) = (a^{1/2}b)a^{1/2} = (a^{1/2}b)(a^{1/2}b)$, so $a^{1/2}b = 0$. It follows that $a^{1/2^n}b = 0$ for all n . Since $\{a^{1/2^n}\} \rightarrow 1$, $b = 0$.

Define $f: X \times Y \rightarrow S$ onto S by $f(x, y) = xy$. The proofs of the following three lemmas are analogous to the proofs in [3].

LEMMA 2. *If $f(x_1, y_1) = f(x_2, y_2) \neq 0$, then either*

- (1) $x_1 = x_2$ and $y_1 = y_2$ or
- (2) $x_1 > x_2$ and $y_2 > y_1$ or
- (3) $x_2 > x_1$ and $y_1 > y_2$.

LEMMA 3. *If $s \in S \setminus \{0\}$, then there exist $(x_1, y_1), (x_2, y_2) \in f^{-1}(s)$ such that for all $(x, y) \in f^{-1}(s)$ we have $x_1 \leq x \leq x_2$ and $y_2 \leq y \leq y_1$.*

LEMMA 4. *If $s \in S \setminus \{0\}$, then $\pi_1(f^{-1}(s))$ is connected.*

LEMMA 5. *If $s \in S \setminus \{0\}$, then $f^{-1}(s)$ is an arc.*

Proof. Let $[x_1, x_2] = \pi_1(f^{-1}(s))$, and define $h: [x_1, x_2] \rightarrow f^{-1}(s)$ by $h(x) = (x, y)$ where y is the unique $y \in Y$ (lemma 2) such that $f(x, y) = s$. Now $h: [x_1, x_2] \rightarrow f^{-1}(s)$ is a continuous, one-to-one, onto function. Thus $h: [x_1, x_2] \rightarrow f^{-1}(s)$ is a homeomorphism, and $f^{-1}(s)$ is an arc.

DEFINITION 6. Let $J = \{(x, y) : (x, y) \in X \times Y \text{ and } f^{-1}(f(x, y)) \text{ is not a point}\}$.

LEMMA 7. *If $s \in f(J)$, then $Xs = sY$.*

The proof of the above lemma is analogous to the proof of Lemma 10 of [3].

LEMMA 8. *If $\{(x, y) : 0 \leq x < x_0, 0 \leq y < y_0\} \subset J$, then $\{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0\} \setminus \{(x_0, y_0)\} \subset J$. Moreover, for each $(x', y') \in \{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0\} \setminus \{(x_0, y_0)\}$ there exists $\bar{x} \in X$ such that $f(\bar{x}, y_0) = f(x', y')$.*

Proof. Let $x_1 \in [0, x_0)$ and fix $x_2 \in (x_1, x_0)$. Then for each $y \in [0, y_0)$

we have $(x_2, y) \in J$. Select an increasing sequence $\{z_n\}$, with $z_n \in [0, y_0]$ and $z_n \rightarrow y_0$. Now there exist $x_3 \in X$ and a sequence $\{w_n\}$, with $w_n \in Y$, such that $x_3x_2 = x_1$, and $x_3f(x_2, z_n) = f(x_2, z_n)w_n$. Now $\{z_nw_n\}$ is an increasing sequence, and hence it must converge. Let $z_nw_n \rightarrow y_1$. Then $f(x_1, y_0) = f(x_2, y_1)$, and $0 \leq y_1 < y_0$. Hence $(x_1, y_0) \in J$. A similar argument shows $(x_0, y_1) \in J$ for $y^1 \in [0, y_0]$.

Next let $(x_1, y_1) \in \{(x, y) : 0 \leq x \leq x_0, 0 \leq x \leq y_0\} \setminus \{(x_0, y_0)\}$. Select $(x_2, y_2) \in \{(x, y) : 0 \leq x \leq x_0, 0 \leq y < y_0\}$ such that $f(x_2, y_2) = f(x_1, y_1)$. Now $(x_2, y_0) \in J$. Fix $y_3 \in J$ such that $y_0y_3 = y_2$. By Lemma 7 there exists $x_3 \in X$ such that $x_3f(x_2, y_0) = f(x_2, y_0)y_3$. Letting $x_4 = x_3x_2$ we have $f(x_4, y_0) = f(x_2, y_2) = f(x_1, y_1)$.

COROLLARY 9. *If $(x, 1), (1, y) \in J$, then $x = 0$ or $y = 0$.*

Proof. Since $(x, 1), (1, y) \in J$ there exist $x_1 \in X, y_1 \in Y$ such that $x_1f(x, 1) = f(x, 1)y$ and $xf(1, y) = f(1, y)y_1$. Thus $x_1x = yy_1$. This is impossible unless $x = 0$ or $y = 0$.

LEMMA 10. *Let $x \in X \setminus \{1\}, y \in Y$. Then yx can be written as $x'y'$ with $x' \in X \setminus \{1\}, y' \in Y$.*

Proof. If $y = 0$ the result is clear. Thus we will assume $y \in Y \setminus \{0\}$. We will divide the proof into several steps.

Step (1). Since $S = Y \cdot X = X \cdot Y$ we know that there exist $x_1 \in X \setminus \{1\}, y_1 \in Y$ such that $y_1x_1 \notin X \cup Y$, and thus there exist $x_2 \in X \setminus \{1\}, y_2 \in Y$ such that $y_1x_1 = x_2y_2$.

Step (2). Let $y_3 \in Y$ with $y_3 \geq y_1$. Then there exists $y_4 \in Y$ such that $y_4y_3 = y_1$. Thus $y_4y_3x_1 = y_1x_1 \notin X \cup Y$. Hence $y_3x_1 \notin Y$.

Step (3). We claim that for $y_3 \in [y_1, 1]$ and n a positive integer, $y_3x_1^{1/n} \notin Y$. For if this were not the case there would exist a positive integer n and a $y_3 \in [y_1, 1]$ such that $y_3x_1^{1/n} = y_6 \in Y$. But by Lemma 2, $y_6 < y_3$. Thus there exists $y_7 \in Y \setminus \{1\}$ such that $y_7y_3 = y_6$. Hence $y_3(x_1^{1/n})^n = y_3x_1^{1/n}(x_1^{1/n})^{n-1} = y_6(x_1^{1/n})^{n-1} = y_7y_3(x_1^{1/n})^{n-1} = \dots = y_7^n y_3 \in Y$. Thus $y_3x_1 \in Y$. This is a contradiction.

Step (4). Let $x \in X \setminus \{1\}$. Then for $y_3 \in [y_1, 1]$ we claim y_3x can be represented as x_8y_8 with $x_8 \in X \setminus \{1\}$, and $y_8 \in Y$. Choose n a positive integer such that $x_1^{1/n} \in [x, 1]$. Then there exists $x_9 \in X$ such that $x_1^{1/n}x_9 = x$. Thus $y_3x = y_3x_1^{1/n}x_9$. However, $y_3x_1^{1/n} \notin Y$, and hence y_3x can be written as x_8y_8 with $x_8 \in X \setminus \{1\}$, and $y_8 \in Y$.

Step (5). Finally, let $x \in X \setminus \{1\}$ and $y \in Y$. If $y = 1$, then $yx = xy$ and $x \in X \setminus \{1\}$ and $y \in Y$. If $y \in Y \setminus \{0, 1\}$, then there exist a positive integer m and $y_3 \in [y_1, 1]$ such that $y = (y_3)^m$. Now $yx = (y_3^m)x = x'y'$ with $x' \in X \setminus \{1\}$, and $y' \in Y$.

The same argument can be used to show that if $x \in X$ and $y \in Y \setminus \{1\}$, then xy can be written as $y'x'$ with $x' \in X$ and $y' \in Y \setminus \{1\}$.

THEOREM 11. *If $s \in S \setminus \{0\}$, then there exist unique $x \in X, y \in Y$ such that $xy = s$.*

Proof. Suppose this is not the case. Then there exist $x_1 \in X \setminus \{0, 1\}, y_1 \in Y \setminus \{0, 1\}$ such that $(x_1, y_1) \in J$. From corollary 9 we can assume $\{(1, y) : y \in Y \setminus \{0\}\} \cap J = \phi$. Let $x_2 = \sup \{x : (x, y_1) \in J\}$. Now $x_2 \in (0, 1)$ and $\{(x, y) : 0 \leq x \leq x_2, 0 \leq y \leq y_1\} \setminus \{(x_2, y_1)\} \subset J$.

Next take $x_3 \in (x_2, 1)$. Then there exist $x_4 \in X \setminus \{0, 1\}, y_4 \in Y$ such that $y_1x_3 = x_4y_4$. If $x_4 \in (0, x_2]$, fix $x_5 \in (x_2, x_3)$. If $x_4 \in (x_2, 1)$, fix $x_5 \in (x_2, \min \{x_3, x_2/x_4\})$ where x_2/x_4 represents the unique element p of X such that $px_4 = x_2$. Take $y_2 \in (y_1, 1)$. Then there exist $x_6 \in X, y_6 \in Y \setminus \{0, 1\}$ such that $y_2x_2 = x_6y_6$. If $y_6 \in (0, y_1]$ fix $y_7 \in (y_1, y_2)$. If $y_6 \in (y_1, 1)$, fix $y_7 \in (y_1, \min \{y_2, y_1/y_6\})$.

For each $x \in [x_2, x_5]$ we have $(xy_1)^2 = x'y'$ with $x' \in (0, x_2]$ and $y' \in (0, y_1]$. By lemma 8 there exists a unique $\bar{x} \in (0, x_2]$ such that $(xy_1)^2 = x'y' = \bar{x}y_1$. Hence we can define a function $x \rightarrow \bar{x}$ from $[x_2, x_5]$ into $(0, x_2]$. The function $x \rightarrow \bar{x}$ defined above is continuous and monotone and thus maps $[x_2, x_5]$ onto an interval $[\bar{x}_2, \bar{x}_5]$.

Also for $y \in [y_1, y_7]$ we have $(x_2y)^2 = \tilde{x}\tilde{y}$ with $\tilde{x} \in (0, x_2]$ and $\tilde{y} \in (0, y_1]$. Again by lemma 8 there exists a unique $x(y) \in (0, x_2]$ such that $(x_2y)^2 = \tilde{x}\tilde{y} = x(y)y_1$. Thus we can define a function $y \rightarrow x(y)$ from $[y_1, y_7]$ into $(0, x_2]$ which is continuous and monotone and hence maps $[y_1, y_7]$ onto an interval $[x(y_1), x(y_7)]$.

Now $(x_2y_1)^2 = \bar{x}_2y_1$ and $(x_2y_1)^2 = x(y_1)y_1$. Hence $\bar{x}_2 = x(y_1)$, so the intervals $[\bar{x}_2, \bar{x}_5]$ and $(x(y_1), x(y_6)]$ intersect. Thus there exist $x \in (x_2, x_5]$ and $y \in (y_1, y_7]$ such that $(xy_1)^2 = (x_2y)^2$. However, $(x, y_1) \notin J$, thus $xy_1 \neq x_2y$. This is a contradiction.

In the same manner we can show that each element $s \in S \setminus \{0\}$ can be written uniquely as yx with $y \in Y$ and $x \in X$.

LEMMA 12. *Let T be a semigroup without zero divisors, $E(T) = \{0, 1\}$, and which is uniquely representable in terms of two usual threads X and Y . Then $T \setminus \{0\}$ is cancellative.*

Proof. Let $s, s_1, s_2 \in T \setminus \{0\}$ with $s = xy, s_1 = x_1y_1, s_2 = x_2y_2$ with $x, x_1, x_2 \in X, y, y_1, y_2 \in Y$, and suppose $ss_1 = ss_2$. Then $xyx_1y_1 = xyx_2y_2$. Now let $yx_1 = \bar{x}_1\bar{y}_1$ and $yx_2 = \bar{x}_2\bar{y}_2$. Thus $x\bar{x}_1\bar{y}_1y_1 = x\bar{x}_2\bar{y}_2y_2$. Since T is uniquely representable we get that $\bar{x}_1 = \bar{x}_2$ and thus $x_1 = x_2$. This implies $\bar{y}_1 = \bar{y}_2$ and hence $y_1 = y_2$. Hence $s_1 = s_2$. In the same manner we can show that if $s, s_1, s_2 \in T \setminus \{0\}$ with $s_1s = s_2s$, then $s_1 = s_2$. Thus

$T \setminus \{0\}$ is cancellative.

COROLLARY 13. *If S is a uniquely divisible semigroup on the two-cell with $E(S) = \{0, 1\}$, then $S \setminus \{0\}$ is a cancellative semigroup.*

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