UNIQUELY REPRESENTABLE SEMIGROUPS ON THE TWO-CELL

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A semigroup $S$ is said to be uniquely representable in terms of two subsets $X$ and $Y$ of $S$ if $X \cdot Y = Y \cdot X = S$, $x_1y_1 = x_2y_2$ is a nonzero element of $S$ implies $x_1 = x_2$ and $y_1 = y_2$, and $y_1x_1 = y_2x_2$ is a nonzero element of $S$ implies $y_1 = y_2$ and $x_1 = x_2$ for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. A semigroup $S$ is said to be uniquely divisible if for each $s \in S$ and every positive integer $n$ there exists a unique $z \in S$ such that $z^n = s$. Theorem. If $S$ is a uniquely divisible semigroup on the two-cell with the set of idempotents of $S$ being a zero for $S$ and an identity for $S$, then $S$ is uniquely representable in terms of $X$ and $Y$ where $X$ and $Y$ are isomorphic copies of the usual unit interval and the boundary of $S$ equals $X$ union $Y$. Corollary. If $S$ is a uniquely divisible semigroup on the two-cell and if $S$ has only two idempotents, a zero and an identity, then the nonzero elements of $S$ form a cancellative semigroup.

A semigroup $S$ is said to be uniquely representable in terms of two subsets $X$ and $Y$ of $S$ if $X \cdot Y = Y \cdot X = S$, $x_1y_1 = x_2y_2$ is a nonzero element of $S$ implies $x_1 = x_2$ and $y_1 = y_2$, and $y_1x_1 = y_2x_2$ is a nonzero element of $S$ implies $y_1 = y_2$ and $x_1 = x_2$ for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. A semigroup $S$ is said to be uniquely divisible if for every $s \in S$ and every positive integer $n$ there exists a unique $z \in S$ such that $z^n = s$.

The primary purpose of this paper is to show that if $S$ is a uniquely divisible semigroup on the two-cell with the set of idempotents of $S$ being a zero for $S$ and an identity for $S$, then $S$ is uniquely representable in terms of $X$ and $Y$ where $X$ and $Y$ are isomorphic copies of the usual unit interval and the boundary of $S$ equals $X$ union $Y$. As a corollary to this theorem we shall prove a conjecture of D. R. Brown, that if $S$ is a uniquely divisible semigroup on the two-cell and $S$ has only two idempotents, a zero and an identity, then the nonzero elements of $S$ form a cancellative subsemigroup of $S$.

Notation. Throughout $S$ will be a uniquely divisible semigroup on the two-cell with $E(S)$ (the set of idempotents of $S$) = \{0, 1\} where 0 is the zero for $S$ and 1 is the identity for $S$. It is well known that the boundary of $S$ is the union of two usual threads $X$ and $Y$ with $X \cap Y = \{0, 1\}$ and $S = X \cdot Y = Y \cdot X$. Intervals containing $x$ will represent segments of $X$ and intervals with $y$ shall stand for segments of $Y$. For a positive integer $n$, $s^{1/n}$ will denote the unique $n$th root of $s$ in $S$. 

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The authors would like to thank the referee for pointing out the following result due to J. D. Lawson and M. Friedberg and which appears in [2].

**Lemma 1.** If $T$ is a uniquely divisible semigroup with $E(T) = \{0, 1\}$, then $T$ has no zero divisors.

**Proof.** Suppose $ab = 0$ for some $a, b \in T$, $a \neq 0$. Then $(ba)^2 = b(ab)a = 0$, hence $ba = 0$. Thus $0 = ab = a^{1/2}(a^{1/2}b) = (a^{1/2}b)a^{1/2} = (a^{1/2}b)(a^{1/2}b)$, so $a^{1/2}b = 0$. It follows that $a^{1/2^n}b = 0$ for all $n$. Since $\{a^{1/2}\} \rightarrow 1$, $b = 0$.

Define $f : X \times Y \to S$ onto $S$ by $f(x, y) = xy$. The proofs of the following three lemmas are analogous to the proofs in [3].

**Lemma 2.** If $f(x_1, y_1) = f(x_2, y_2) \neq 0$, then either
1. $x_1 = x_2$ and $y_1 = y_2$ or
2. $x_1 > x_2$ and $y_2 > y_1$ or
3. $x_2 > x_1$ and $y_1 > y_2$.

**Lemma 3.** If $s \in S \setminus \{0\}$, then there exist $(x_1, y_1), (x_2, y_2) \in f^{-1}(s)$ such that for all $(x, y) \in f^{-1}(s)$ we have $x_1 \geq x \geq x_2$ and $y_2 \geq y \geq y_1$.

**Lemma 4.** If $s \in S \setminus \{0\}$, then $\pi_1(f^{-1}(s))$ is connected.

**Lemma 5.** If $s \in S \setminus \{0\}$, then $f^{-1}(s)$ is an arc.

**Proof.** Let $[x_1, x_2] = \pi_1(f^{-1}(s))$, and define $h : [x_1, x_2] \to f^{-1}(s)$ by $h(x) = (x, y)$ where $y$ is the unique $y \in Y$ (lemma 2) such that $f(x, y) = s$. Now $h : [x_1, x_2] \to f^{-1}(s)$ is a continuous, one-to-one, onto function. Thus $h : [x_1, x_2] \to f^{-1}(s)$ is a homeomorphism, and $f^{-1}(s)$ is an arc.

**Definition 6.** Let $J = \{(x, y) : (x, y) \in X \times Y$ and $f^{-1}(f(x, y))$ is not a point$\}.$

**Lemma 7.** If $s \in f(J)$, then $Xs = sY$.

The proof of the above lemma is analogous to the proof of Lemma 10 of [3].

**Lemma 8.** If $\{(x, y) : 0 \leq x < x_0, 0 \leq y < y_0\} \subset J$, then $\{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0\} \subset J$. Moreover, for each $(x', y') \in \{(x, y) : 0 \leq x \leq x_0, 0 \leq y \leq y_0\}$ there exists $x \in X$ such that $f(x, y_0) = f(x', y)$.

**Proof.** Let $x_1 \in [0, x_0]$ and fix $x_2 \in (x_1, x_0)$. Then for each $y \in [0, y_0)$
we have \((x_2, y) \in J\). Select an increasing sequence \(\{z_n\}\), with \(z_n \in [0, y_0)\) and \(z_n \to y\). Now there exist \(x_3 \in X\) and a sequence \(\{w_n\}\), with \(w_n \in Y\), such that \(x_3 f(x_2, z_n) = f(x_2, z_n) w_n\). Now \(\{z_n w_n\}\) is an increasing sequence, and hence it must converge. Let \(z_n w_n \to y_1\). Then 
\[f(x_3, y_0) = f(x_2, y_1),\]
and \(0 \leq y_1 < y_0\). Hence \((x_1, y_0) \in J\). A similar argument shows \((x_0, y_1) \in J\) for \(y_1 \in [0, y_0)\).

Next let \((x, y_0) \in J\) there exist \(x_1 \in X, y_1 \in Y\) such that \(x, f(x, y_0) = f(x, y_1)\). Now \(x_2 = x_3 x_2\) we have 
\[f(x_4, y_0) = f(x_2, y_0) = f(x_3, y_0) x_2 = f(x_2, y_0) y_3 = f(x_1, y_1)\].

**Corollary 9.** If \((x, 1), (1, y) \in J\), then \(x = 0\) or \(y = 0\).

**Proof.** Since \((x, 1), (1, y) \in J\) there exist \(x_1 \in X, y_1 \in Y\) such that 
\[x_1 f(x, 1) = f(x, 1) y\]
and 
\[x f(1, y) = f(1, y) y_1\]. Thus \(x_2 x = y y_1\). This is impossible unless \(x = 0\) or \(y = 0\).

**Lemma 10.** Let \(x \in X \setminus \{1\}, y \in Y\). Then \(x y\) can be written as \(x' y'\) with \(x' \in X \setminus \{1\}, y' \in Y\).

**Proof.** If \(y = 0\) the result is clear. Thus we will assume \(y \in Y \setminus \{0\}\). We will divide the proof into several steps.

Step (1). Since \(S = Y \cdot X = X \cdot Y\) we know that there exist \(x_1 \in X \setminus \{1\}, y_1 \in Y\) such that \(y_1 x_1 \in X \cup Y\), and thus there exist \(x_2 \in X \setminus \{1\}, y_2 \in Y\) such that \(y_1 x_1 = x_2 y_2\).

Step (2). Let \(y_3 \in Y\) with \(y_3 \geq y_1\). Then there exists \(y_4 \in Y\) such that 
\[y_4 y_3 x_1 = y_4 x_1 \in X \cup Y\]. Hence \(y_3 x_1 \in Y\).

Step (3). We claim that for \(y_1 \in [y_1, 1]\) and \(n\) a positive integer, 
\[y_n x_1 = \in Y\]. For if this were not the case there would exist a positive integer \(n\) and a \(y_5 \in [y_1, 1]\) such that \(y_n x_1 = y_5 \in Y\). But by Lemma 2, \(y_5 < y_1\). Thus there exists \(y_7 \in Y \setminus \{1\}\) such that \(y_7 y_3 = y_5\). Hence 
\[y_3 (x_1)^n = y_3 x_1 (x_1)^n y_3 = y_3 (x_1)^n y_3 = \cdots = y_5 y_3 \in Y\].
Thus \(y_3 x_1 \in Y\). This is a contradiction.

Step (4). Let \(x \in X \setminus \{1\}\). Then for \(y_5 \in [y_1, 1]\) we claim \(y_5 x\) can be represented as \(x_5 y_5\) with \(x_5 \in X \setminus \{1\}\), and \(y_5 \in Y\). Choose \(n\) a positive integer such that \(x_1^n \in [x, 1]\). Then there exists \(x_6 \in X\) such that 
\[x_6 x_1^n x_6 = x\]. Thus \(y_5 x = y_5 x_1^n x_6\). However, \(y_5 x_1^n \in Y\), and hence \(y_5 x\) can be written as \(x_5 y_5\) with \(x_5 \in X \setminus \{1\}\), and \(y_5 \in Y\).

Step (5). Finally, let \(x \in X \setminus \{1\}\) and \(y \in Y\). If \(y = 1\), then \(y x = x y\) and \(x \in X \setminus \{1\}\) and \(y \in Y\). If \(y \in Y \setminus \{0, 1\}\), then there exist a positive integer \(m\) and \(y_5 \in [y_1, 1]\) such that \(y = y_5^m\). Now \(y x = (y_5^m x = x' y')\) with \(x' \in X \setminus \{1\}\), and \(y' \in Y\).
The same argument can be used to show that if \( x \in X \) and \( y \in Y \setminus \{1\} \), then \( xy \) can be written as \( y'x' \) with \( x' \in X \) and \( y' \in Y \setminus \{1\} \).

**Theorem 11.** If \( s \in S \setminus \{0\} \), then there exist unique \( x \in X \), \( y \in Y \) such that \( xy = s \).

**Proof.** Suppose this is not the case. Then there exist \( x_1 \in X \setminus \{0, 1\} \), \( y_1 \in Y \setminus \{0, 1\} \) such that \( (x_1, y_1) \in J \). From corollary 9 we can assume \( \{1, y\} \cap J = \emptyset \). Let \( x_3 = \sup \{x: (x, y_i) \in J\} \). Now \( x_3 \in (0, 1) \) and \( \{x(y): 0 \leq x \leq x_3, 0 \leq y \leq y_i\} \cap J \).

Next take \( x_5 \in (x_3, 1) \). Then there exist \( x_4 \in X \setminus \{0, 1\} \), \( y_4 \in Y \) such that \( y_4x_4 = x_5y_4 \). If \( x_4 \in (0, x_4) \), fix \( x_5 \in (x_3, x_4) \). If \( x_4 \in (x_3, 1) \), fix \( x_5 \in (x_3, 1) \) where \( x_5/x_4 \) represents the unique element \( p \) of \( X \) such that \( px_4 = x_5 \). Take \( y_7 \in (y_4, 1) \). Then there exist \( y_6 \in X \), \( y_7 \in Y \setminus \{0, 1\} \) such that \( y_6x_6 = x_7y_6 \). If \( y_6 \in (0, y_6] \) fix \( y_7 \in (y_4, y_6] \). If \( y_6 \in (y_4, 1) \), fix \( y_7 \in (y_4, 1) \) such that \( y_7x_7 = x_7y_7 \).

For each \( x \in [x_3, x_4] \) we have \( (xy)^2 = x'y' \) with \( x' \in (0, x_3] \) and \( y' \in (0, y_4] \). By lemma 8 there exists a unique \( \bar{x} \in (0, x_3] \) such that \( (x'y')^2 = x'y' = \bar{x}y \). Hence we can define a function \( x \mapsto \bar{x} \) from \( [x_3, x_4] \) into \( (0, x_3] \). The function \( x \mapsto \bar{x} \) defined above is continuous and monotone and thus maps \( [x_3, x_4] \) onto an interval \( [\bar{x}_3, \bar{x}_4] \).

Also for \( y \in [y_4, y_5] \) we have \( (x'y')^2 = x'y' \) with \( \bar{x} \in (0, x_3] \) and \( y' \in (0, y_5] \). Again by lemma 8 there exists a unique \( x(y) \in (0, x_3] \) such that \( (x'y')^2 = \bar{x}y = x(y)y \). Thus we can define a function \( y \mapsto x(y) \) from \( [y_4, y_5] \) into \( (0, x_3] \) which is continuous and monotone and hence maps \( [y_4, y_5] \) onto an interval \( [x(y_4), x(y_5)] \).

Now \( (x'y')^2 = \bar{x}y \) and \( (x'y')^2 = x(y)y \). Hence \( \bar{x} = x(y) \), so the intervals \( (\bar{x}_3, \bar{x}_4] \) and \( (x(y), x(y_5)] \) intersect. Thus there exist \( x \in [x_3, x_4] \) and \( y \in [y_4, y_5] \) such that \( (x'y')^2 = (x'y')^2 \). However, \( (x, y_i) \notin J \), \( xy_1 = x'_2y_1 \). This is a contradiction.

In the same manner we can show that each element \( s \in S \setminus \{0\} \) can be written uniquely as \( yx \) with \( y \in Y \) and \( x \in X \).

**Lemma 12.** Let \( T \) be a semigroup without zero divisors, \( E(T) = \{0, 1\} \), and which is uniquely representable in terms of two usual threads \( X \) and \( Y \). Then \( T \setminus \{0\} \) is cancellative.

**Proof.** Let \( s, s_1, s_2 \in T \setminus \{0\} \) with \( s = xy, s_1 = x_1y_1, s_2 = x_2y_2 \) with \( x, x_1, x_2 \in X \), \( y, y_1, y_2 \in Y \), and suppose \( ss_1 = ss_2 \). Then \( xyx_2y_1 = xyx_2y_2 \). However, \( (x, y_i) \notin J \), \( x_1 = x_2 \). This implies \( y_1 = y_2 \) and hence \( y_1 = y_2 \). Hence \( s_1 = s_2 \). In the same manner we can show that if \( s, s_1, s_2 \in T \setminus \{0\} \) with \( s_1s = s_2s \), then \( s_1 = s_2 \). Thus
$T\setminus\{0\}$ is cancellative.

**Corollary 13.** If $S$ is a uniquely divisible semigroup on the two-cell with $E(S) = \{0, 1\}$, then $S\setminus\{0\}$ is a cancellative semigroup.

**References**


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