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CHARACTERIZATIONS OF UNIFORM CONVEXITY

WILLIAM LEE BYNUM

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In this paper, three new characterizations of uniform convexity of a Banach space X are established. The characterization developed in Theorem 1 resembles the definition of the modulus of smoothness given by J. Lindenstrauss. The characterizations developed in Theorems 2 and 3 are interrelated, both involving the duality map of X into X^* . The methods used are adapted to give an abbreviated proof of a recent result of W. V. Petryshyn relating the strict convexity of X to the duality map of X into X^* .

The following definitions are included for reference. For a Banach space X , the *unit sphere* of X , denoted by S_1 , is the set of all elements of X having norm 1. A Banach space X is *uniformly convex* if for each t in $(0, 2]$, $2 - \delta(t) = \inf \{2 - \|x + y\| : x, y \in S_1, \|x - y\| \geq t\}$ is positive ([1], [2]) (the function δ is called the *modulus of convexity* of X). A direct consequence of this definition is that each of the following conditions is equivalent to X being uniformly convex:

(i) Whenever $\{a_n\}$ and $\{b_n\}$ are sequences in S_1 such that $\|a_n + b_n\| \rightarrow 2$, then $\|a_n - b_n\| \rightarrow 0$.

(ii) Whenever $\{a_n\}$ and $\{b_n\}$ are sequences in X such that $\|a_n\| \rightarrow 1$, $\|b_n\| \rightarrow 1$, and $\|a_n + b_n\| \rightarrow 2$, then $\|a_n - b_n\| \rightarrow 0$. (see [3, p. 113] or [9, p. 109]). The *modulus of smoothness* of X is the function ρ such that for $t \geq 0$,

$$2 \rho(t) = \sup \{\|x + ty\| + \|x - ty\| - 2 : x, y \in S_1\}$$

([5]). A Banach space X is *strictly convex* if for each x and y in S_1 such that $x \neq y$ and each λ in $(0, 1)$, $\|\lambda x + (1 - \lambda)y\| < 1$ ([1], [6]). A function $J: X \rightarrow 2^{X^*}$ is a *duality map* of X into X^* if for each x in X , $J(x) = \{w \in X^* : (w, x) = w(x) = \|w\| \|x\| \text{ and } \|w\| = \|x\|\}$ (see [6] for notation and a list of pertinent literature).

I would like to thank Professor Tosio Kato for suggesting the following formulation of Theorem 1.

THEOREM 1. *Let ϕ be a strictly convex and strictly increasing function on $[0, 2]$ such that $\phi(1) = 1$. Then X is uniformly convex if and only if for each t in $(0, 1)$, $\alpha(t) = \inf \{\phi(\|x + ty\|) + \phi(\|x - ty\|) - 2 : x, y \in S_1\}$ is positive.*

Proof. Suppose that X is uniformly convex and that there is a t in $(0, 1]$ such that $\alpha(t) = 0$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in S_1 such that if we let $a_n = x_n + ty_n$ and $b_n = x_n - ty_n$, then $\phi(\|a_n\|) + \phi(\|b_n\|) \rightarrow 2$. Since ϕ is convex and nondecreasing and $\phi(1) = 1$, $2 \leq 2\phi((\|a_n\| + \|b_n\|)/2) \leq \phi(\|a_n\|) + \phi(\|b_n\|) \rightarrow 2$ and thus by the strict convexity of ϕ , we have that $|\|a_n\| - \|b_n\|| \rightarrow 0$. The preceding inequality and the continuity of ϕ^{-1} at 1 imply that $\|a_n\| + \|b_n\| \rightarrow 2$ and consequently that $\|a_n\| \rightarrow 1$ and $\|b_n\| \rightarrow 1$. For each n , $\|a_n + b_n\| = 2$, so the uniform convexity of X implies that $2t = \|a_n - b_n\| \rightarrow 0$, which is contradictory.

Now suppose that $\alpha(t)$ is positive for each t in $(0, 1]$. For fixed x and y in S_1 , the function $h(t) = \phi(\|x + ty\|) + \phi(\|x - ty\|) - 2$ is convex, and since $h(0) = 0$ and $h \geq 0$ on $[0, 1]$, h is nondecreasing. Therefore, since α is the infimum of a collection of nondecreasing functions, α is nondecreasing on $[0, 1]$. By the definition of α , if $\|y\| \leq \|x\| \neq 0$, then $\phi(\|x + y\|/\|x\|) + \phi(\|x - y\|/\|x\|) - 2 \geq \alpha(\|y\|/\|x\|)$. Thus if a and b are in S_1 and $\|a - b\| \leq \|a + b\|$, we have that

$$(1) \quad 2\phi(2/\|a + b\|) - 2 \geq \alpha(\|a - b\|/\|a + b\|) \geq \alpha(\|a - b\|/2).$$

Now, let $\{a_n\}$ and $\{b_n\}$ be sequences in S_1 such that $\|a_n + b_n\| \rightarrow 2$. We may assume that for sufficiently large n , $\|a_n - b_n\| \leq \|a_n + b_n\|$. Thus inequality (1) and the continuity of ϕ at 1 imply that $\alpha(\|a_n - b_n\|/2) \rightarrow 0$, so $\|a_n - b_n\| \rightarrow 0$ and X is uniformly convex.

Inequality (1) above gives a bound on the modulus of convexity, δ , in terms of ϕ^{-1} and α . By considering each of the cases $\|a - b\| \leq \|a + b\|$, $1 \leq \|a + b\| \leq \|a - b\|$, and $\|a + b\| < 1$, it follows that $2\delta(\|a - b\|)$ is not less than the smaller of

$$1, \|a - b\| \{\phi^{-1}(1 + 1/2 \alpha(1/2)) - 1\},$$

and $\|a - b\| \{\phi^{-1}(1 + 1/2 \alpha(\|a - b\|/2)) - 1\}$.

In Theorem 1, the case when $\phi(t) = t^2$ merits special attention. Note that for each Banach space X and each t in $[0, 1]$, $\alpha(t) \leq 2t^2$; moreover, X is an inner product space if and only if $\alpha(t) = 2t^2$ for each t in $[0, 1]$. In the same vein, note that X obeys a weak parallelogram law (i.e., there is a λ in $(0, 1]$ such that for each x and y in X , $\|x + y\|^2 + \lambda \|x - y\|^2 \leq 2 \|x\|^2 + 2 \|y\|^2$ - see [4]) if and only if there is a μ in $(0, 2]$ such that $\alpha(t) \geq \mu t^2$ for each t in $[0, 1]$.

THEOREM 2. *A Banach space X is uniformly convex if and only if for each t in $(0, 2]$, $\beta(t) = \inf \{1 - (f, y) : x, y \in S_1, \|x - y\| \geq t, f \in J(x)\}$ is positive, where J is the duality map from X into X^* .*

Proof. If X is uniformly convex and $x, y \in S_1$ and $f \in J(x)$, then

$$1 - (f, y) = 2 - (f, x + y) \geq 2 - \|x + y\| \geq 2\delta(\|x - y\|).$$

Now suppose that $\beta > 0$ on $(0, 2]$ and that X is not uniformly convex. Then by the definition there exist sequences $\{x_n\}$ and $\{y_n\}$ in S_1 such that $0 < \|x_n + y_n\| \rightarrow 2$ and for each $n, \|x_n - y_n\| \geq t$. For each n , let $a_n = \|x_n + y_n\|^{-1}, z_n = a_n(x_n + y_n), h_n \in J(z_n), f_n \in J(x_n)$, and $g_n \in J(y_n)$. Then,

$$2 - \|x_n + y_n\| = 1 - (h_n, x_n) + 1 - (h_n, y_n) \geq \beta(\|x_n - z_n\|) + \beta(\|y_n - z_n\|).$$

But neither $\|x_n - z_n\|$ nor $\|y_n - z_n\|$ is less than $ta_n - |1 - 2a_n|$, so that for sufficiently large n , we have $\|x_n - z_n\| \geq t/4, \|y_n - z_n\| \geq t/4$, and $2 - \|x_n + y_n\| \geq 2\beta(t/4)$, which is contradictory.

THEOREM 3. *A Banach space X is uniformly convex if and only if the duality map J of X into X^* is uniformly monotone in the sense that for each t in $(0, 2], \gamma(t) = \inf \{(f - g, x - y) : x, y \in S_1, \|x - y\| \geq t, f \in J(x), g \in J(y)\}$ is positive.*

Proof. If X is uniformly convex and $x, y \in S_1, f \in J(x), g \in J(y)$, then $(f - g, x - y) = 2 - (g, x + y) + 2 - (f, x + y) \geq 2(2 - \|x + y\|)$, so J is uniformly monotone.

Suppose J is uniformly monotone and X is not uniformly convex. By Theorem 2, $\beta(t) = 0$ for some t in $(0, 2]$; i.e., there exist sequences $\{x_n\}$ and $\{y_n\}$ in S_1 and $\{f_n\}$ in X^* such that for each n ,

$$f_n \in J(x_n), \|x_n - y_n\| \geq t,$$

and $1 - (f_n, y_n) \rightarrow 0$. Since $1 - (f_n, y_n) \geq 2 - \|x_n + y_n\| \geq 0$, then $\|x_n + y_n\| \rightarrow 2$ and we may assume that $\|x_n + y_n\| > 0$ for each n . As in Theorem 2, let $a_n = \|x_n + y_n\|^{-1}, z_n = a_n(x_n + y_n)$, and $h_n \in J(z_n)$. Thus, $(h_n, x_n + y_n) = \|x_n + y_n\| \rightarrow 2$ and since $\|h_n\| = 1 = \|x_n\| = \|y_n\|$, then $(h_n, x_n) \rightarrow 1$. So,

$$(h_n - f_n, z_n - x_n) = 1 - a_n - a_n(f_n, y_n) + 1 - (h_n, x_n) \rightarrow 0.$$

However, as in Theorem 2, for sufficiently large n , we have that $\|x_n - z_n\| \geq t/4$ and $(h_n - f_n, z_n - x_n) \geq \gamma(t/4)$, which is contradictory.

Now we turn to the previously mentioned result of Petryshyn [6, Theorem 1, p. 284-287]. We need the following theorem, proved in slightly different form in [8, Theorem, part iii]. We include a proof of it here for completeness. In the sequel, we shall use the following characterization of strict convexity due to Ruston [7]: A Banach space X is strictly convex if and only if for x and y in S_1 such that $x \neq y, 2 - \|x + y\| > 0$.

Theorem (Torrance [8]). A Banach space X is strictly convex if and only if for x and y in S_1 such that $x \neq y$ and for f in $J(x)$, $1 - (f, y) > 0$.

Proof. Suppose that X is strictly convex and let x, y , and f be as above. Then, $1 - (f, y) \geq 2 - \|x + y\| > 0$.

Now suppose that the second condition of the theorem is satisfied and that X is not strictly convex. Then, there exist $x, y \in S_1 (x \neq y)$ such that $\|x + y\| = 2$. Let $z = (x + y)/2$ and $h \in J(z)$. Since $\|h\| = 1 = \|x\| = \|y\|$ and $(h, x + y) = 2, (h, x) = 1$, a contradiction, since $z \neq x$.

Theorem (Petryshyn [6]). A Banach space X is strictly convex if and only if the duality map J of X into X^* is *strictly monotone* in the sense that if $x \neq y, f \in J(x)$, and $g \in J(y)$, then $(f - g, x - y) > 0$.

Proof. Suppose that X is strictly convex. Let $x, y \in X, f \in J(x)$, and $g \in J(y)$. Then, $\|f\| \|y\| - (f, y) \geq \|f\| (\|x\| + \|y\| - \|x + y\|)$ and $\|g\| \|x\| - (g, x) \geq \|g\| (\|x\| + \|y\| - \|x + y\|)$ and by the use of equation (#) of [6], we have

$$(f - g, x - y) \geq (\|x\| - \|y\|)^2 + (\|x\| + \|y\|)(\|x\| + \|y\| - \|x + y\|).$$

If $x \neq y$ and $\|x\| = \|y\|$, then $\|x\| > 0$ and $\|x\| + \|y\| - \|x + y\| = \|x\| (2 - \|x/\|x\| + y/\|x\|)$, which is positive by the strict convexity of X . Consequently, J is strictly monotone.

Now, suppose that J is strictly monotone and that X is not strictly convex. Then by the previous theorem, there exist $x, y \in S_1 (x \neq y)$ and an $f \in J(x)$ such that $1 - (f, y) = 0$. As before, $1 - (f, y) \geq 2 - \|x + y\|$, so $\|x + y\| = 2$. If $z = (x + y)/2$ and $h \in J(z)$, then $(h, x + y) = 2$ and $\|h\| = 1 = \|x\| = \|y\|$, so $(h, x) = 1$. Consequently, $(h - f, z - x) = 1 - (h, x) + 1 - (f, z) = 0$, which contradicts the fact that $z \neq x$.

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