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**A CONTINUOUS FORM OF SCHWARZ'S LEMMA IN NORMED
LINEAR SPACES**

LAWRENCE ALBERT HARRIS

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Our main result is an inequality which shows that a holomorphic function mapping the open unit ball of one normed linear space into the closed unit ball of another is close to being a linear map when the Fréchet derivative of the function at 0 is close to being a surjective isometry. We deduce this result as a corollary of a kind of uniform rotundity at the identity of the sup norm on bounded holomorphic functions mapping the open unit ball of a normed linear space into the same space.

Let Δ be the open unit disc of the complex plane, and let $f: \Delta \rightarrow \bar{\Delta}$ be a holomorphic function with $f(0) = 0$. It is easy to show that the inequality

$$(1) \quad |f(z) - f'(0)z| \leq \frac{2|z|^2}{1-|z|} (1 - |f'(0)|)$$

holds for all $z \in \Delta$. (For example, apply the lemma given in [5] to the function $z^{-1}f(z)$. See also [3, §292].) Qualitatively, inequality (1) means that if $f'(0)$ is close to the unit circle then $f(z)$ is close to being a linear function of z as long as z remains a fixed positive distance away from the exterior of the unit disc. Our purpose is to prove a version of (1) which applies to vector-valued holomorphic functions of vectors. We deduce this result from an extremal inequality for holomorphic functions, which reduces to a theorem of G. Lumer in the linear case. It should be pointed out that the inequalities we obtain cannot be proved simply by composing with linear functionals and applying the 1-dimensional case, as for instance the generalized Cauchy inequalities can.

1. **Main results.** In the following, a function h defined on an open subset of a complex normed linear space with range in another is called *holomorphic* if the Fréchet derivative of h at x (denoted by $Dh(x)$) exists as a bounded complex-linear map for each x in the domain of definition of h . (See [7, Def. 3.16.4].) Denote the open (resp., closed) unit ball of a normed linear space X by X_0 (resp., X_1). Throughout, X and Y denote arbitrary complex normed linear spaces. Our main result is

THEOREM 1. *Let $h: X_0 \rightarrow Y_1$ be a holomorphic function with*

$h(0) = 0$. Put $L = Dh(0)$ and let \mathcal{U} be the set of all linear isometries of X onto Y . Suppose \mathcal{U} is nonempty and let $d(L, \mathcal{U})$ denote the distance of L from \mathcal{U} in the operator norm.

Then

$$\|h(x) - L(x)\| \leq \frac{8 \|x\|^2}{(1 - \|x\|)^2} d(L, \mathcal{U}), \quad (x \in X_0).$$

Clearly Theorem 1 contains the main result of [5], i.e., $h = L$ when L is in \mathcal{U} . In fact, it is a consequence of Theorem 1 that any sequence of holomorphic functions $h_n: X_0 \rightarrow Y_1$ converges uniformly to a linear map L in \mathcal{U} on closed subballs of X_0 whenever the sequence of derivatives $Dh_n(0)$ converges to L in the operator norm. This may be proved by showing as in [5] that $h_n(0) \rightarrow 0$, and then applying Theorem 1 to the function $(1 + \|h_n(0)\|)^{-1} [h_n(x) - h_n(0)]$.

Let I be the identity map on X and let the symbol $\|\cdot\|$, when applied to functions, denote the supremum over X_0 . We deduce Theorem 1 from

THEOREM 2. *Let $\delta \geq 0$ and suppose $h: X_0 \rightarrow X$ is a holomorphic function satisfying*

$$(2) \quad \|I + \lambda h\| \leq 1 + \delta$$

for all $\lambda \in \bar{D}$. Let P_m be the m th term of the Taylor series expansion for h about 0. Then

$$(3) \quad \|P_m\| \leq K_m \delta,$$

where $K_0 = 1$, $K_1 = e$ and $K_m = m^{m/(m-1)}$, $m \geq 2$. If inequality (2) holds when the values of λ are restricted to ± 1 , then (3) still holds but with δ replaced by $\sqrt{\delta(2 + \delta)}$.

Recall that by definition

$$(4) \quad P_m(x) = \frac{1}{m!} \left[\frac{d^m}{d\lambda^m} h(\lambda x) \right]_{\lambda=0}, \quad P_0(x) = h(0).$$

Hence $P_1 = Dh(0)$. Moreover [7, Th. 26.3.6], P_m is of the form $P_m(x) = F_m(x, \dots, x)$, where F_m is a continuous symmetric m -linear map. It should be noted that in general P_m is a mapping of X into the completion of X .

2. *Proof of Theorem 1 assuming Theorem 2.* Let $h: X_0 \rightarrow X_1$ be a holomorphic function with $h(0) = 0$ and put $L = Dh(0)$. It suffices to prove that h satisfies the inequality

$$(5) \quad \|h(x) - L(x)\| \leq \frac{8 \|x\|^2}{(1 - \|x\|)^2} \|I - L\|, \quad (x \in X_0);$$

for Theorem 1 can then be deduced by composing the given function with inverses of linear maps in \mathcal{L} and applying (5). Thus to prove (5), let

$$h(x) = P_1(x) + P_2(x) + \dots, (P_1 = L),$$

be the Taylor series expansion for h about 0. This series converges to $h(x)$ for every x in X_0 . (See [7, pp. 109-113].) Let $x \in X_1$ and let ℓ be a linear functional on the completion of X with $\|\ell\| \leq 1$. Define $f(\lambda) = \ell(\lambda^{-1}h(\lambda x))$. Then $f: \Delta \rightarrow \bar{\Delta}$ is holomorphic and

$$f(\lambda) = \sum_{m=0}^{\infty} a_m \lambda^m, a_m = \ell(P_{m+1}(x)).$$

By [9, p. 172], we have $|a_{m-1}| \leq 1 - |a_0|^2 \leq 2(1 - |a_0|)$ for $m \geq 2$, and hence

$$\left| \ell(L(x) + \frac{1}{2} \lambda P_m(x)) \right| \leq 1$$

for all $\lambda \in \Delta$. It follows from the Hahn-Banach Theorem that $\|L + 1/2 \lambda P_m\| \leq 1$, and therefore

$$\left\| I + \frac{1}{2} \lambda P_m \right\| \leq 1 + \delta, \delta = \|I - L\|,$$

for all $\lambda \in \Delta$. Since P_m extends to the completion of X , Theorem 2 applies to show that

$$\|P_m\| \leq 2K_m \delta \leq 8(m - 1)\delta,$$

where the last inequality follows from the inequalities $m/(m - 1) \leq 2$ and $m \leq 2^{m-1}$. Hence if $x \in X_0$,

$$\|h(x) - L(x)\| \leq \sum_{m=2}^{\infty} \|P_m(x)\| \leq \frac{8 \|x\|^2 \delta}{(1 - \|x\|)^2},$$

which is (5).

3. *Proof of Theorem 2.* Our proof is an elaboration of an iteration argument due to H. Cartan. (See [1, pp. 13-14].) Clearly we may suppose that $\delta > 0$ and that inequality (2) is strict. Let N be any positive integer satisfying $N \geq 1/\delta$ and put $r = 1/(N\delta)$. Then by the triangle inequality,

$$(6) \quad \|I + \lambda rh\| = \|(1 - r)I + r(I + \lambda h)\| < 1 + 1/N$$

for all $\lambda \in \Delta$. Take $\alpha = (1 + 1/N)^{-1}$. Our strategy is to compute the derivatives with respect to λ of the n th iterate of the function $\alpha I + \lambda \alpha rh$ and then apply the generalized Cauchy inequalities [7, p. 97]. The number n of iterations we take will depend on N .

Let $x \in X_0$ and define

$$f_n(\lambda) = (\alpha I + \lambda \alpha r h)^n(x) .$$

By (6), $f_n: \Delta \rightarrow X$ is a well-defined holomorphic function satisfying

$$(7) \quad \|f_n(\lambda)\| < 1, (\lambda \in \Delta) .$$

Clearly $f'_1(0) = \alpha r h(x)$, and differentiating the identity

$$f_{n+1}(\lambda) = \alpha f_n(\lambda) + \lambda \alpha r h(f_n(\lambda)) ,$$

we have

$$f'_{n+1}(0) = \alpha f'_n(0) + \alpha r h(\alpha^n x) .$$

Therefore, by induction

$$(8) \quad f'_n(0) = \sum_{k=0}^{n-1} \alpha^{n-k} r h(\alpha^k x) .$$

By (7) and Cauchy's inequality,

$$(9) \quad \|f'_n(0)\| \leq 1 .$$

Let $\Phi_n(x)$ be the right hand side of (8). Clearly each Φ_n is holomorphic in X_0 and by (9), $\|\Phi_n\| \leq 1$. Applying the Cauchy inequalities, we have

$$\left\| \frac{1}{m!} \left[\frac{d^m}{d\lambda^m} \Phi_n(\lambda x) \right]_{\lambda=0} \right\| \leq 1 , \quad (x \in X_0) .$$

Hence by (4),

$$(10) \quad \left\| \sum_{k=0}^{n-1} \alpha^{n+(m-1)k} r P_m(x) \right\| \leq 1 , \quad (x \in X_0) ,$$

so

$$\|P_m\| \leq \frac{1 - \alpha^{m-1}}{r \alpha^n [1 - \alpha^{n(m-1)}]} ,$$

assuming $m \geq 2$. Since $1 - \alpha^{m-1} \leq (m - 1)(1 - \alpha)$, $1/r = N\delta$ and $N(1/\alpha - 1) = 1$, it follows that

$$(11) \quad \|P_m\| \leq \frac{(m - 1)\delta}{\alpha^{n-1}[1 - \alpha^{n(m-1)}]} .$$

Finally, letting n be the greatest integer in $N(m - 1)^{-1} \log m$ and taking the limit in (11) as $N \rightarrow \infty$, we obtain inequality (3) for $m \geq 2$. When $m = 1$, inequality (3) follows from (10) with $n = N$. When $m = 0$, we may obtain (3) from (9) by letting $x = 0$ and taking the limit as $n \rightarrow \infty$.

The proof of the second part of Theorem 2 follows from quite

general considerations. Suppose $\|I \pm h\| \leq 1 + \delta$. By the first part of Theorem 2, it suffices to prove that the inequality

$$(12) \quad \|I + \lambda th\| \leq 1 + t^2, \quad t = \sqrt{\delta(2 + \delta)},$$

holds for all $\lambda \in \Delta$. To do this, let $x \in X_0$ and $\varphi \in (X^*)_1$ be given. Then $|\varphi(x) \pm \varphi(h(x))| \leq 1 + \delta$, and consequently $|\varphi(x)|^2 + |\varphi(h(x))|^2 \leq (1 + \delta)^2$. Hence if $\lambda \in \Delta$, $|\varphi(x + \lambda th(x))| \leq |\varphi(x)| + t |\varphi(h(x))|$

$$\leq (1 + t^2)^{1/2}(1 + \delta) = 1 + t^2,$$

where the last inequality follows from the Cauchy-Schwarz inequality. This in conjunction with the Hahn-Banach Theorem proves (12).

4. Further remarks. Note that by Theorem 2 (or by [2, §§2, 3]) if $\delta \geq 0$ and $L: X \rightarrow X$ is a linear map satisfying $\|I \pm L\| \leq 1 + \delta$, then $\|L\| \leq e\sqrt{\delta(2 + \delta)}$. This readily implies Theorem 18 of [8]. Note also that in the case $\delta = 0$, Theorem 2 shows that I is an extreme point of $H^\infty(X_0, X)_1$, where $H^\infty(X_0, X)$ denotes the space of all bounded holomorphic functions $h: X_0 \rightarrow X$ with the sup norm. A simpler proof of this fact has already been given in [6]. It would be interesting to know whether or not K_m is the best possible constant in (3) which is independent of δ and h . See [4] for a related result.

Note added in proof. The author has recently shown that the answer to the above is affirmative.

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