A CONTINUOUS FORM OF SCHWARZ’S LEMMA IN NORMED LINEAR SPACES

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Our main result is an inequality which shows that a holomorphic function mapping the open unit ball of one normed linear space into the closed unit ball of another is close to being a linear map when the Fréchet derivative of the function at 0 is close to being a surjective isometry. We deduce this result as a corollary of a kind of uniform rotundity at the identity of the sup norm on bounded holomorphic functions mapping the open unit ball of a normed linear space into the same space.

Let \( \Delta \) be the open unit disc of the complex plane, and let \( f: \Delta \to \Delta \) be a holomorphic function with \( f(0) = 0 \). It is easy to show that the inequality

\[
|f(z) - f'(0)z| \leq \frac{2|z|^2}{1 - |z|}(1 - |f'(0)|)
\]

holds for all \( z \in \Delta \). (For example, apply the lemma given in [5] to the function \( z^{-1}f(z) \). See also [3, §292].) Qualitatively, inequality (1) means that if \( f'(0) \) is close to the unit circle then \( f(z) \) is close to being a linear function of \( z \) as long as \( z \) remains a fixed positive distance away from the exterior of the unit disc. Our purpose is to prove a version of (1) which applies to vector-valued holomorphic functions of vectors. We deduce this result from an extremal inequality for holomorphic functions, which reduces to a theorem of G. Lumer in the linear case. It should be pointed out that the inequalities we obtain cannot be proved simply by composing with linear functionals and applying the 1-dimensional case, as for instance the generalized Cauchy inequalities can.

1. Main results. In the following, a function \( h \) defined on an open subset of a complex normed linear space with range in another is called holomorphic if the Fréchet derivative of \( h \) at \( x \) (denoted by \( Dh(x) \)) exists as a bounded complex-linear map for each \( x \) in the domain of definition of \( h \). (See [7, Def. 3.16.4].) Denote the open (resp., closed) unit ball of a normed linear space \( X \) by \( X_0 \) (resp., \( X_1 \)). Throughout, \( X \) and \( Y \) denote arbitrary complex normed linear spaces. Our main result is

**Theorem 1.** Let \( h: X_0 \to Y_1 \) be a holomorphic function with
\( h(0) = 0. \) Put \( L = Dh(0) \) and let \( \mathcal{Z} \) be the set of all linear isometries of \( X \) onto \( Y. \) Suppose \( \mathcal{Z} \) is nonempty and let \( d(L, \mathcal{Z}) \) denote the distance of \( L \) from \( \mathcal{Z} \) in the operator norm. Then

\[
\| h(x) - L(x) \| \leq \frac{8 \| x \|^2}{(1 - \| x \|^2)^2} d(L, \mathcal{Z}), \quad (x \in X_0).
\]

Clearly Theorem 1 contains the main result of [5], i.e., \( h = L \) when \( L \) is in \( \mathcal{Z}. \) In fact, it is a consequence of Theorem 1 that any sequence of holomorphic functions \( h_n : X_0 \to Y_1 \) converges uniformly to a linear map \( L \) in \( \mathcal{Z} \) on closed subballs of \( X_0 \) whenever the sequence of derivatives \( Dh_n(0) \) converges to \( L \) in the operator norm. This may be proved by showing as in [5] that \( h_n(0) \to 0, \) and then applying Theorem 1 to the function \( (1 + \| h_n(0) \|)^{-\frac{1}{2}} [h_n(x) - h_n(0)]. \)

Let \( I \) be the identity map on \( X \) and let the symbol \( \| \| \), when applied to functions, denote the supremum over \( X_0. \) We deduce Theorem 1 from

**Theorem 2.** Let \( \delta \geq 0 \) and suppose \( h : X_0 \to X \) is a holomorphic function satisfying

\[
(2) \quad \| I + \lambda h \| \leq 1 + \delta
\]

for all \( \lambda \in \mathbb{J}. \) Let \( P_m \) be the \( m \)th term of the Taylor series expansion for \( h \) about \( 0. \) Then

\[
(3) \quad \| P_m \| \leq K_m \delta,
\]

where \( K_0 = 1, \ K_1 = e \) and \( K_m = m^{m/(m-1)}, \) \( m \geq 2. \) If inequality (2) holds when the values of \( \lambda \) are restricted to \( \pm 1, \) then (3) still holds but with \( \delta \) replaced by \( \sqrt{\delta(2 + \delta)}. \)

Recall that by definition

\[
(4) \quad P_m(x) = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} h(\lambda x) \right]_{\lambda=0}, \quad P_0(x) = h(0).
\]

Hence \( P_1 = Dh(0). \) Moreover [7, Th. 26.3.6], \( P_m \) is of the form \( P_m(x) = F_m(x, \ldots, x), \) where \( F_m \) is a continuous symmetric \( m \)-linear map. It should be noted that in general \( P_m \) is a mapping of \( X \) into the completion of \( X. \)

2. **Proof of Theorem 1 assuming Theorem 2.** Let \( h : X_0 \to X_1 \) be a holomorphic function with \( h(0) = 0 \) and put \( L = Dh(0). \) It suffices to prove that \( h \) satisfies the inequality

\[
(5) \quad \| h(x) - L(x) \| \leq \frac{8 \| x \|^2}{(1 - \| x \|^2)^2} \| I - L \|, \quad (x \in X_0);
\]
for Theorem 1 can then be deduced by composing the given function with inverses of linear maps in \( \mathcal{C} \) and applying (5). Thus to prove (5), let
\[
h(x) = P_1(x) + P_2(x) + \cdots, \quad (P_i = L),
\]
be the Taylor series expansion for \( h(x) \) about 0. This series converges to \( h(x) \) for every \( x \) in \( X \). (See [7, pp. 109-113].) Let \( x \in X \) and let \( \mathcal{L} \) be a linear functional on the completion of \( X \) with \( ||\mathcal{L}|| \leq 1 \). Define \( f(\lambda) = \mathcal{L}(\lambda^{-1}h(\lambda x)) \). Then \( f: \mathcal{A} \to \mathcal{A} \) is holomorphic and
\[
f(\lambda) = \sum_{m=0}^{\infty} a_m \lambda^m, \quad a_m = \mathcal{L}(P_{m+1}(x)).
\]
By [9, p. 172], we have \( |a_{m-1}| \leq 1 - |a_0|^2 \leq 2(1 - |a_0|) \) for \( m \geq 2 \), and hence
\[
|\mathcal{L}(L(x) + \frac{1}{2} \lambda P_m(x))| \leq 1
\]
for all \( \lambda \in \mathcal{A} \). It follows from the Hahn-Banach Theorem that \( ||L + 1/2 \lambda P_m|| \leq 1 \), and therefore
\[
||I + \frac{1}{2} \lambda P_m|| \leq 1 + \delta, \quad \delta = ||I - L||,
\]
for all \( \lambda \in \mathcal{A} \). Since \( P_m \) extends to the completion of \( X \), Theorem 2 applies to show that
\[
||P_m|| \leq 2K_m \delta \leq 8(m - 1)\delta,
\]
where the last inequality follows from the inequalities \( m/(m - 1) \leq 2 \) and \( m \leq 2^{n-1} \). Hence if \( x \in X \),
\[
||h(x) - L(x)|| = \sum_{m=2}^{\infty} ||P_m(x)|| \leq \frac{8 ||x|| \delta}{(1 - ||x||)^2},
\]
which is (5).

3. Proof of Theorem 2. Our proof is an elaboration of an iteration argument due to H. Cartan. (See [1, pp. 13-14].) Clearly we may suppose that \( \delta > 0 \) and that inequality (2) is strict. Let \( N \) be any positive integer satisfying \( N \geq 1/\delta \) and put \( r = 1/(N\delta) \). Then by the triangle inequality,
\[
||I + \lambda rh|| = ||(1 - r)I + r(I + \lambda h)|| < 1 + 1/N
\]
for all \( \lambda \in \mathcal{A} \). Take \( \alpha = (1 + 1/N)^{-1} \). Our strategy is to compute the derivatives with respect to \( \lambda \) of the nth iterate of the function \( \alpha I + \lambda x rh \) and then apply the generalized Cauchy inequalities [7, p. 97]. The number \( n \) of iterations we take will depend on \( N \).
Let $x \in X_0$ and define

$$f_n(\lambda) = (\alpha I + \lambda \alpha r h)^n(x).$$

By (6), $f_n: \mathcal{A} \to X$ is a well-defined holomorphic function satisfying

(7) $$\|f_n(\lambda)\| < 1, \quad (\lambda \in \mathcal{A}).$$

Clearly $f'_1(0) = \alpha r h(x)$, and differentiating the identity

$$f_{n+1}(\lambda) = \alpha f_n(\lambda) + \lambda \alpha r h(f_n(\lambda)),$$

we have

$$f'_{n+1}(0) = \alpha f'_n(0) + \alpha r h(\alpha^n x).$$

Therefore, by induction

(8) $$f'_n(0) = \sum_{k=0}^{n-1} \alpha^{n-k} r h(\alpha^k x).$$

By (7) and Cauchy's inequality,

(9) $$\|f'_n(0)\| \leq 1.$$

Let $\Phi_n(x)$ be the right hand side of (8). Clearly each $\Phi_n$ is holomorphic in $X_0$ and by (9), $\|\Phi_n\| \leq 1$. Applying the Cauchy inequalities, we have

$$\left| \frac{1}{m!} \left[ \frac{d^n}{d\lambda^n} \Phi_n(\lambda x) \right]_{\lambda=0} \right| \leq 1, \quad (x \in X_0).$$

Hence by (4),

(10) $$\left| \sum_{k=0}^{\infty} \alpha^{n+(m-1)k} r P_n(x) \right| \leq 1, \quad (x \in X_0),$$

so

$$\|P_n\| \leq \frac{1 - \alpha^{m-1}}{r \alpha^m [1 - \alpha^{(m-1)}]},$$

assuming $m \geq 2$. Since $1 - \alpha^{m-1} \leq (m - 1)(1 - \alpha)$, $1/r = N\delta$ and $N(1/\alpha - 1) = 1$, it follows that

(11) $$\|P_n\| \leq \frac{(m - 1)\delta}{\alpha^{m-1} [1 - \alpha^{(m-1)}]}.$$

Finally, letting $n$ be the greatest integer in $N(m - 1)^{-1} \log m$ and taking the limit in (11) as $N \to \infty$, we obtain inequality (3) for $m \geq 2$. When $m = 1$, inequality (3) follows from (10) with $n = N$. When $m = 0$, we may obtain (3) from (9) by letting $x = 0$ and taking the limit as $n \to \infty$.

The proof of the second part of Theorem 2 follows from quite
general considerations. Suppose $||I \pm h|| \leq 1 + \delta$. By the first part of Theorem 2, it suffices to prove that the inequality

$$(12) \quad ||I + \lambda th|| \leq 1 + t^2, \quad t = \sqrt{\delta(2 + \delta)},$$

holds for all $\lambda \in A$. To do this, let $x \in X_0$ and $\sigma \in (X^*)$, be given. Then $|\sigma(x) \pm \sigma(h(x))| \leq 1 + \delta$, and consequently $|\sigma(x)|^2 + |\sigma(h(x))|^2 \leq (1 + \delta)^2$. Hence if $\lambda \in A$, $|\sigma(x + \lambda th(x))| \leq |\sigma(x)| + t |\sigma(h(x))|

$$\leq (1 + t^2)^{1/2}(1 + \delta) = 1 + t^2,$$

where the last inequality follows from the Cauchy-Schwarz inequality. This in conjunction with the Hahn-Banach Theorem proves (12).

4. Further remarks. Note that by Theorem 2 (or by [2, §§2, 3]) if $\delta \geq 0$ and $L: X \to X$ is a linear map satisfying $||I \pm L|| \leq 1 + \delta$, then $||L|| \leq e \sqrt{\delta(2 + \delta)}$. This readily implies Theorem 18 of [8]. Note also that in the case $\delta = 0$, Theorem 2 shows that $I$ is an extreme point of $H^\infty(X_0, X)$, where $H^\infty(X_0, X)$ denotes the space of all bounded holomorphic functions $h: X_0 \to X$ with the sup norm. A simpler proof of this fact has already been given in [6]. It would be interesting to know whether or not $K_m$ is the best possible constant in (3) which is independent of $\delta$ and $h$. See [4] for a related result.

Note added in proof. The author has recently shown that the answer to the above is affirmative.

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